Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 15
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QUIZ 1

Your Name:

You can have one single-sided handwritten sheet of paper with anything you want on it. You can write your solutions on this exam.

1. Consider the following linear program.

$$
\begin{array}{lrrrl}
\max & 3x_1 & +2x_2 & +4x_3 & \\
s.t. & 2x_1 & -x_2 & -3x_3 & \leq 10 \\
 & x_1 & -2x_2 & +3x_3 & \leq 7 \\
 & -x_1 & +2x_2 & +4x_3 & \leq 12 \\
 & x_1 & +x_2 & +x_3 & \leq 11 \\
 & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0 &
\end{array}
$$

   Write its dual.

2. (a) Give the smallest matrix A with entries in $\{-1, 0, 1\}$ that is not totally unimodular.

(b) Is the following true?

> If a general graph does not have a perfect matching then there exists a set S of vertices whose removal creates at least $|S| + 1$ odd connected components.

True or **False**.

If false, give a brief explanation.

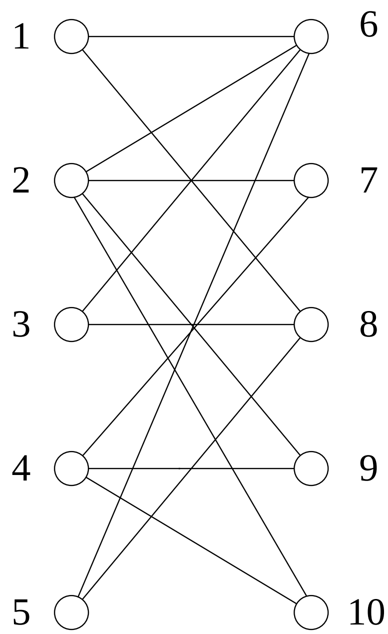
(c) Is the following true?

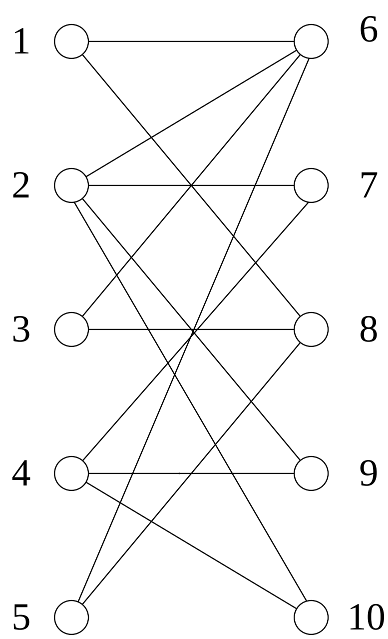
> If, for some $b \in \mathbb{Z}^m$, $\{x \in \mathbb{R}^n : Ax \leq b\}$ has only integer-valued vertices then A is totally unimodular.

True or **False**.

If false, give a brief explanation.

3. Find a minimum vertex cover C in the following graph (list the vertices in C). Argue why it is optimum.





Job j	d_j
1	5
2	1
3	4
4	3
5	3
6	2
7	2
8	5
9	2
10	3

4. Consider the following scheduling problem. We are given 10 jobs. Each job requires one unit of processing time. All jobs are available at time 0, and job j has a profit of j units (e.g job 7 has a profit of 7 units) and a deadline d_j. The profit for job j will only be earned if the job completes by time d_j.

   (a) To solve this problem efficiently, one can define a matroid $M = (E, \mathcal{I})$ where $E = \{1, 2, \cdots, 10\}$. What is $\mathcal{I}$? (You do not need to argue that M is a matroid.)

Job j	d_j
1	5
2	1
3	4
4	3
5	3
6	2
7	2
8	5
9	2
10	3

(b) Give a set of jobs of maximum profit which can all be scheduled on time. Explain briefly how you found this. What is this maximum profit?

(c) Give one base of M.

Job j	d_j
1	5
2	1
3	4
4	3
5	3
6	2
7	2
8	5
9	2
10	3

(d) Give one circuit of M.

(e) What is $span(\{4, 5, 7\})$?

(f) (Bonus.) Is this matroid M representable over $\mathbb{R}$? (No lengthy justification is needed.)

5. We are given an undirected graph $G = (V, E)$ and integer values $p(v)$ for every vertex $v \in V$. We would like to know if we can orient the edges of G such that the directed graph we obtain has at most $p(v)$ arcs incoming to v (the "indegree requirements"). In other words, for each edge $\{u, v\}$, we have to decide whether to orient it as (u, v) or as (v, u), and we would like at most $p(v)$ arcs oriented towards v.

   (a) Formulate the problem of deciding whether an orientation exists as a bipartite matching problem on a graph $H = (A \cup B, F)$ where one side (say A) of the bipartition corresponds to E and the other side (say B) will have $\sum_v p(v)$ vertices. Specify what the edges of this graph H are, and also what the existence (or non-existence) of an orientation meeting the indegree requirements means in terms of bipartite matchings in this graph. (No justification is needed.)

(b) State Hall's theorem giving a necessary and sufficient condition for a bipartite graph $H = (A \cup B, F)$ to have a matching of size $|A|$.

(c) Derive from Hall's theorem on this bipartite graph H a necessary and sufficient condition for the existence of an orientation of G satisfying the indegree requirements.