Solution to 4-12

First, let us prove that the conditions are sufficient.

Consider two independent set I_1 and I_2 such that (i) holds. Let f be the only element in $I_2 \setminus I_1$, and consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 1, & \text{if } e \in I_1, \\ 0, & \text{if } e = f, \\ -1, & \text{if } e \notin I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2. Therefore I_1 and I_2 are adjacent. The case where (ii) holds is analogous.

Now, assume that I_1 and I_2 satisfy (iii). For this case let f be the only element in $I_2 \setminus I_1$ and g be the only element in $I_1 \setminus I_2$. Consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 2, & \text{if } e \in I_1 \cap I_2, \\ 1, & \text{if } e = f, \text{ or } e = g \\ -1, & \text{if } e \notin I_1 \cup I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2, and so they are adjacent in the matroid polytope.

Now let us prove that the conditions are necessary.

Assume that I_1 and I_2 are a pair of adjacent independent sets and let $c : E \to \mathbb{R}$ be a cost function that is maximized only by I_1 and I_2. In particular note that $c(e) \geq 0$ for every element in $I_1 \cup I_2$. Assume w.l.o.g. that $|I_1| \leq |I_2|$.

Case 1:$|I_2| > |I_1|$. By the exchange axiom $(I3)$, there exists an element $f \in I_2 \setminus I_1$ such that $I_1 + f$ is an independent set and, by a previous observation, it has weight greater or equal than the weight of I_1. Since I_1 is optimum it follows that so is $I_1 + f$. Since I_2 and I_1 are the only optimums, it follows that $I_2 = I_1 + f$. Therefore, (i) holds.

Case 2: $|I_2| = |I_1|$. Let f be the element in $I_1 \Delta I_2 = I_1 \setminus I_2 \cup I_2 \setminus I_1$ with minimum cost. Assume w.l.o.g. that $f \in I_1$. Clearly, $I_1 - f$ is an independent set and $|I_1 - f| < |I_2|$. It follows that there exists an element $g \in I_2 \setminus I_1$ such that $I_1 - f + g$ is an independent set. By choice of f, $c(I_1 - f + g) = c(I_1) - c(f) + c(g) \geq c(I_1)$. But then $I_1 - f + g$ is also a maximum weight independent set. Since I_2 and I_1 were the only optimums, it follows that $I_2 = I_1 - f + g$, which implies that $|I_2 \setminus I_1| = |I_2 \setminus I_2| = 1$.

To conclude that (iii) holds, we only need to show that $I_1 \cup I_2 \notin \mathcal{I}$. But this is easy to see since, in other case, using that $c(e) \geq 0$ for every $e \in I_1 \cup I_2$, we would have that $c(I_1 \cup I_2) \geq c(I_1)$. This implies that $I_1 \cup I_2$ is another optimum (different from I_1 and I_2), which contradicts the adjacency condition of I_1 and I_2.