Problem set 6

This problem set is due in class on Monday April 28th.

1. Suppose you are given an $m \times n$ matrix $A \in \mathbb{R}^{m\times n}$ with row sums $r_1, \cdots, r_m \in \mathbb{Z}$ and column sums $c_1, \cdots, c_n \in \mathbb{Z}$. Some of the entries might not be integral but the row sums and column sums are. Show that there exists a rounded matrix A' with the following properties:

   - row sums and column sums of A and A' are identical,
   - $a'_{ij} = \lceil a_{ij} \rceil$ or $a'_{ij} = \lfloor a_{ij} \rfloor$ (i.e. a'_{ij} is a_{ij} either rounded up or down.).

   (Hint. Think of flows.)

   By the way, this rounding is useful when one would like to publish statistics without giving too much information on specific individuals (for example for census purposes). One would then want to modify the entries (not necessarily to a neighboring integer, but to say a neighboring multiple of 1000) without modifying row and column sums.

2. At some point during baseball season, each of n teams of the American League has already played several games. Suppose team i has won w_i games so far, and $g_{ij} = g_{ji}$ is the number of games that teams i and j have yet to play. No game ends in a tie, so each game gives one point to either team and 0 to the other. You would like to decide if your favorite team (Red Sox?), say team n, can still win. In other words, you would like to determine whether there exists an outcome to the games to be played (remember, with no ties) such that team n has at least as many victories as all the other teams (we allow team n to be tied for first place with other teams).

   Show that this problem can be solved as a maximum flow problem. Explain.

3. Consider the $s - t$ flow problem on a directed graph in which every directed edge has a lower bound $l(e) = 0$.

   (a) Write the dual linear program of this maximum flow problem.

   (To make the dual a bit nicer to interpret, it is useful to add a variable $f \in \mathbb{R}$ in the primal, impose that the net flow of out of s equals f and that the net flow out of t equals $-f$, and maximize f. In the dual linear program, we will have one variable, say y_v, for every vertex $v \in V$ (which corresponds to either the flow balance constraint if $v \notin \{s,t\}$ or the newly introduced constraints if $v \in \{s,t\}$), and also one dual variable, say z_e for every directed edge (arc) $e \in E$ (corresponding to the inequalities $x_e \le u(e)$).)

   (b) Does this dual always have an optimum solution that is integral (even if capacities are not integral)?

(c) Show that for any *integer* solution to the dual (i.e. $y_v, z_e \in \mathbb{Z}$) you can obtain an $s-t$ cut of value at most the value of the dual solution.

(d) Now suppose you are given a non necessarily integral feasible solution for this dual of value V (in the dual linear program). Show how we can obtain a cut of value at most V.

4. Let G be an undirected graph in which the degree of every vertex is at least k. Show that there exist two vertices s and t with at least k edge-disjoint paths between them.

5. (Optional.) We'll consider here another algorithm to find a minimum (global) cut in an undirected graph $G = (V, E)$ with capacities $u : E \to \mathbb{R}_+$. Let $d(\cdot)$ denote the cut function, i.e.

$$d(S) = u(\delta(S)) = \sum_{e \in \delta(S)} u(e),$$

for every set $S \subset V$. Say that an (unordered) pair of vertices $\{u, v\}$ define a *good pair* if $d(S) \geq \min_{x \in S} d(\{x\})$ for every set S separating u and v, i.e. $|S \cap \{u, v\}| = 1$.

(a) Assuming one has an algorithm to find a good pair in a capacitated undirected graph, give an algorithm to find a global min cut in a capacitated undirected graph.

(b) To find a good pair (and show that one always exists), the algorithm uses a different ordering than the maximum adjacency ordering given in lecture. A minimum degree ordering is defined as follows. Suppose we have already selected the first $i-1$ vertices where $1 \leq i \leq n$, and let $A_i = \{v_1, v_2, \cdots, v_{i-1}\}$. For $i = 1$, we have $A_0 = \emptyset$. Now define v_i to be $\arg\min_{v \in V \setminus A_i} u(\{v\} : V \setminus A_i \setminus \{v\})$, i.e. v has minimum total capacity between v and the remaining unselected vertices (thus v_1 has total minimum capacity on edges incident to it). In a minimum degree ordering, (v_{n-1}, v_n) (where $n = |V|$) is not necessarily a pendant pair. Prove that $\{v_{n-1}, v_n\}$ is always a good pair.