Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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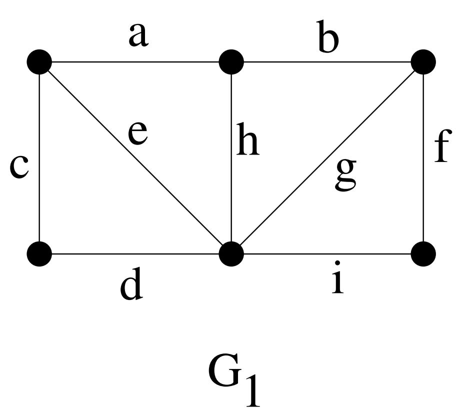
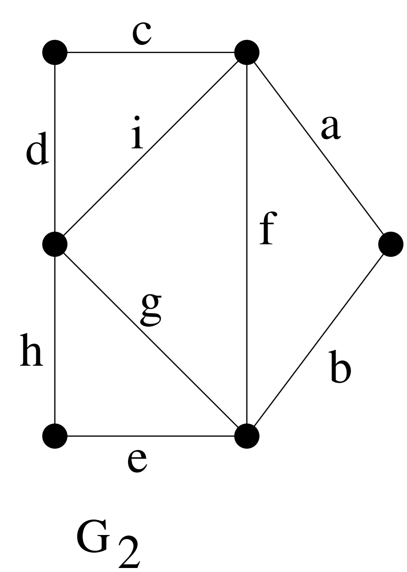
Handout 15
April 12th, 2011

Problem set 5

This problem set is due in class on April 21st, 2011.

1. We are given the following two graphs G_1 and G_2 with edge set $E = \{a, b, c, d, e, f, g, h, i\}$.





   Observe that $S = \{a, b, c, d\}$ is a forest in both G_1 and in G_2, so it is independent in $M_1 = M(G_1)$ and $M_2 = M(G_2)$. Construct the exchange graph corresponding to S, and indicate which elements are sources and sinks. Using the exchange graph, find a larger set of elements which is acyclic in both G_1 and in G_2.

2. Consider a graph $G = (V, E)$ with $|E| = 2(|V| - 1)$ and suppose the edges are partitioned into $|V| - 1$ blue edges (in B) and $|V| - 1$ red edges (in R). Suppose furthermore that G is the union of two edge-disjoint spanning trees (with no restriction on the colors of the edges). Show that G contains a tree with at most $\lceil \frac{|V|-1}{2} \rceil$ blue edges and with at most $\lceil \frac{|V|-1}{2} \rceil$ red edges. (Hint: think polytopes...)

3. Consider the strongly connected subgraph problem. We are given a directed graph $D = (V, A)$, a cost function $c : A \to \mathbb{R}$ and the goal is to find a subset F of arcs such that (V, F) is strongly connected (i.e. for every $u, v \in V$, there exists a directed path from u to v).

   Suppose we adapt the algorithm we saw in lecture for the minimum cost r-arborescence problem to this problem. That means we first modify the linear program by considering all constraints

$$\sum_{a \in \delta^-(S)} x_a \geq 1$$

where now S is any subset of V. As long as the solution is not strongly connected, we select a set S corresponding to a strongly connected component with no (selected) incoming arc, and raise its dual variable. We add the corresponding tight edge and we repeat. When the solution is strongly connected, we consider the arcs we have added in the reverse order they were added and we delete any that is not necessary for the strong connectivity of the current graph.

- Give an instance showing that this algorithm does not necessarily construct a minimum cost strongly connected subgraph.
- If we try to use the same proof as we did for the r-arborescence problem, where does it fail?