Problem set 4

This problem set is due in class on Tuesday March 29th, 2011.

1. Exercise 4-8 from the notes on matroids.

2. Exercise 4-12 from the notes on matroids.

3. (Bonus.) Let $G = (V, E)$ be a graph. Define a set F of edges to be *independent* if F can be partitioned into F_1 and F_2 where both F_1 and F_2 are forests. Show that this notion of independence defines a matroid. (This is probably hard with what we have seen in class so far.)

4. We are given an undirected graph $G = (V, E)$ and integer values $p(v)$ for every vertex $v \in V$. We would like to know if we can orient the edges of G such that the directed graph we obtain has at most $p(v)$ arcs incoming to v (the "indegree requirements"). In other words, for each edge $\{u, v\}$, we have to decide whether to orient it as (u, v) or as (v, u), and we would like at most $p(v)$ arcs oriented towards v.

   (a) Formulate the problem of deciding whether an orientation exists as a bipartite matching problem on a graph $H = (A \cup B, F)$ where one side (say A) of the bipartition corresponds to E and the other side (say B) will have $\sum_v p(v)$ vertices. Specify what the edges of this graph H are, and also what the existence (or non-existence) of an orientation meeting the indegree requirements means in terms of bipartite matchings in this graph.

   (b) State Hall's theorem giving a necessary and sufficient condition for a bipartite graph $H = (A \cup B, F)$ to have a matching of size $|A|$.

   (c) Derive from Hall's theorem on this bipartite graph H a necessary and sufficient condition for the existence of an orientation of G satisfying the indegree requirements.