FINAL

This exam is closed book. You are allowed to one double-sided handwritten sheet of paper with notes. Be neat! Enjoy the summer!

Your Name:

1. For each of the questions below, give an answer and a one-line justification.

   (a) In general, is it possible for the projection P' of a polyhedron P to have more vertices than P itself?

   (b) And how about for the number of faces? Can a projection P' of P have more faces than P?

   (c) Given an *undirected* graph $G = (V, E)$ with capacities $u : E \rightarrow \mathbb{R}_+$, let λ_{uv} be the capacity of a minimum cut separating u and v. What is the maximum number of distinct λ_{uv} over all graphs and capacities with n vertices, i.e. how large can $|\{\lambda_{uv} : u, v \in V, u \neq v\}|$ be as a function of $n = |V|$?

(d) Consider a directed graph $G = (V, E)$ with upper capacities $u : E \to \mathbb{R}_+$ and lower capacities equal to 0. Let $f(s, t)$ be the maximum flow value from s to t. Is the following TRUE or FALSE? For all $r, s, t \in V$: $\min(f(r, s), f(s, t)) \leq f(r, t)$.

(e) Suppose we would like to find the minimum cost spanning tree (or a minimum cost base in a matroid) using local search (i.e. moving from one spanning tree to a smallest cost one in its neighborbood). Do you know of any exact neighborhood of polynomial size (in the number of vertices, edges or size of the ground set)?

(f) Is the following TRUE or FALSE? In a (not necessarily bipartite) graph, the size of a maximum matching is at most the size of a minimum vertex cover.

2. Consider the polyhedron P:

$$P = \{x : \begin{array}{rrrrl} x_1 & +2x_2 & +3x_3 & +4x_4 & \leq 8 \\ & 3x_2 & -2x_3 & +2x_4 & \leq 0 \\ 2x_1 & +3x_2 & -x_3 & +5x_4 & \leq 6 \\ & x_2 & +x_3 & & \leq 3 \\ & & 4x_3 & +x_4 & \leq 5 \\ -2x_1 & & -3x_3 & +5x_4 & \leq 0 \\ & & x_3 & +x_4 & \leq 3 \\ & & & x_4 & \leq 1\} \end{array}$$

Let P_1 be the projection of P onto $\{(x_2, x_3, x_4)\}$, i.e. $P_1 = \{(x_2, x_3, x_4) : \exists x_1$ with $(x_1, x_2, x_3, x_4) \in P\}$. Describe P_1 as a system of linear inequalities.

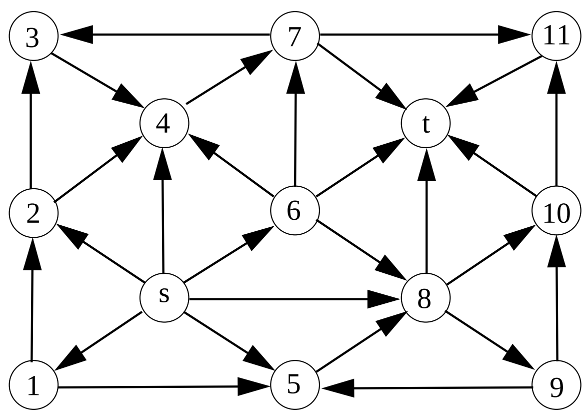
Is $x = (x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ an extreme point of P? Justify.

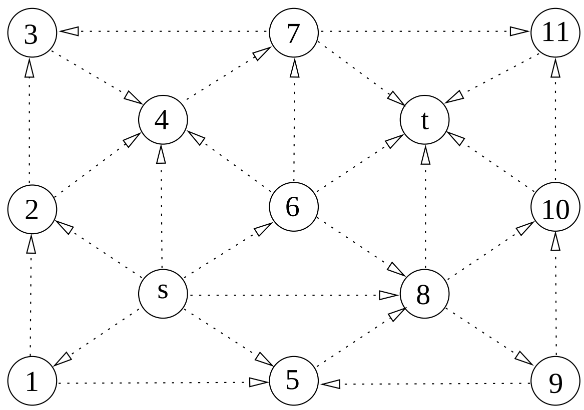
$$
P = \{x : \begin{array}{rrrrl}
x_1 & +2x_2 & +3x_3 & +4x_4 & \leq 8 \\
 & 3x_2 & -2x_3 & +2x_4 & \leq 0 \\
2x_1 & +3x_2 & -x_3 & +5x_4 & \leq 6 \\
 & x_2 & +x_3 & & \leq 3 \\
 & & 4x_3 & +x_4 & \leq 5 \\
-2x_1 & & -3x_3 & +5x_4 & \leq 0 \\
 & & x_3 & +x_4 & \leq 3 \\
 & & & x_4 & \leq 1\}
\end{array}
$$

3. Suppose we are given a graph $G = (V, E)$ with capacities $u : E \to \mathbb{R}_+$ and a set $T \subseteq V$ of even cardinality. Give the high-level description of an algorithm for finding a minimum T-odd cut in this graph. State a key theorem (without proof) regarding the proof of correctness of your algorithm (something more descriptive than just: **Theorem:** This algorithm is correct. :-)

4. State Menger's theorem giving a min-max relation involving the maximum number of arc-disjoint paths between s and t in a directed graph.

   In the directed graph below, find a maximum number of arcs disjoint paths between s and t, and provide a short proof of optimality. Write explicitly the vertices along each of your paths.

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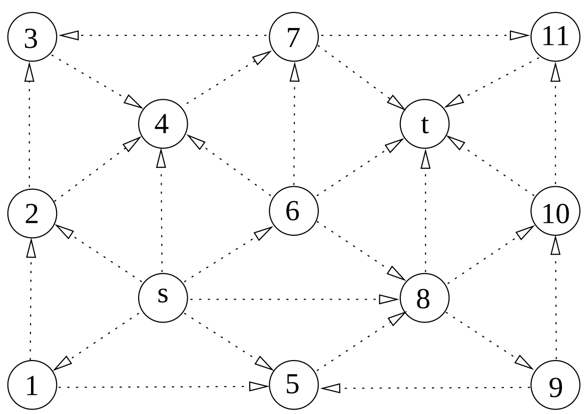
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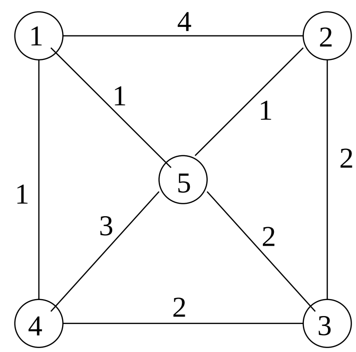
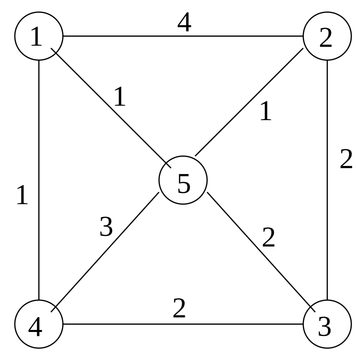
5. Consider the following undirected graph; edge capacities are indicated along every edge. Find the cut of minimum capacity. Show your work. You can use any algorithm that you would also use if the instance had 1000 vertices (so complete enumeration is out of the question).

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6. Consider a (not necessarily bipartite) graph $G = (V, E)$ and a profit function $p : V \to \mathbb{R}_+$. (The profit function is defined on the vertices of G.) Our goal is to find a matching M which maximizes $\sum_{v \in V(M)} p(v)$ where $V(M)$ denotes the vertices matched in M. How would you solve this problem? Justify your solution.

7. Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) (so $V = A \cup B$). Suppose we are also given a matroid $M = (A, \mathcal{I})$ defined on A (one of the sides of the bipartition). We would like to restrict our attention to *independent matchings* M which are those matchings M such that $\{a \in A : \exists b \in B \text{ with } (a, b) \in M\} \in \mathcal{I}$. The maximum independent matching problem is the problem of finding an independent matching of maximum cardinality. How can this problem be solved efficiently? Justify your answer. (You can use as building blocks things we have seen in lectures, anything else needs to be justified.)

Can you give a complete description of the convex hull of (incidence vectors of) independent matchings in terms of linear inequalities (your description does not need to be minimal)? (This is a polytope in $\mathbb{R}^{|E|}$.)