Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Solutions to Problem Set 4

1. Suppose that a partition matroid is specified by $E = E_1 \sqcup E_2 \cdots \sqcup E_l$ and $\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq k_i, i = 1, \cdots, l\}$, where $k_i \leq |E_i|$. For each $i = 1, \dots, l$, consider the sub matroid $M_i = (E_i, \mathcal{I}_i)$, where $\mathcal{I}_i = \{X \subseteq E_i : |X| \leq k_i\}$. Clearly if A_i is a matrix that represents M_i over R, then the block diagonal matrix $A_1 \bigoplus \cdots \bigoplus A_l$ represents M over $\mathbb{R}$.

   It is easy to see that each M_i is the uniform matroid $U_{k_i,|E_i|}$.

   Now, we only need to show that how we can represent the uniform matroid $U_{k,n}$ over $\mathbb{R}$. Take $v_1, \dots, v_n \in \mathbb{R}^k$, where $v_i = (1, i, i^2, \dots, i^{k-1})$. Then any subset of $\{v_1, \cdots, v_n\}$ of cardinality k is independent due to the Vandermonde determinant. On the other hand, any subset of size greater than k must be dependent since we are in $\mathbb{R}^k$. So the matrix whose columns are $v_1, \dots, v_n$ represents the uniform matroid $U_{k,n}$.

   Therefore, the partition matroid M, represents by block diagonal matrix, whose i-th block is isomorphic to the matrix representing $U_{k_i,|E_i|}$.

2. It is easy to see that the first axiom is satisfied. For the second, consider the following bipartite graph: one set of vertices is $A = \cup A_i$ and the other is $B = \{A_i : 1 \leq i \leq n\}$. A pair of vertices $a \in A$ and $A_i \in B$ forms an edge iff $a \in A_i$. There is a correspondence between partial transversals of the family of sets and matchings of the graph: given a matching, the corresponding partial transversal is given by all the vertices in A that have an edge of the matching incident to them. Consider two transversals X, Y such that $|X| < |Y|$, with associated matchings with edges M and N. Consider the sets $Y \setminus X$ and $X \setminus Y$ and alternating paths in $M \triangle N$ starting at vertices in $Y \setminus X$. Because of the cardinality condition we have that $|Y \setminus X| > |X \setminus Y|$, which implies that not all such alternating paths can end at vertices in $X \setminus Y$, that is, there exists $a \in Y \setminus X$ such that the alternating path in $M \triangle N$ starting from it ends at a vertex in B. Then $X \cup \{a\}$ is also a partial transversal, as we can augment the matching M by means of the alternating path starting at a.

3. It is easy to see that $(E, \mathcal{I})$ satisfies the first axiom (I_1) that if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$. For (I_2), consider $X, Y \in \mathcal{I}$ and $|Y| > |X|$, in order to show the second axiom (I_2), we need to show that there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. Let us call a set $S \in \mathcal{F}$ maximal in $T \subseteq E$, $T \neq S$, if $S \subset T$ and S is not contained in any other element of $\mathcal{F}$ that is properly contained in T. Suppose that $A_1, \dots, A_n$ are the maximal sets in E. Set $A^* = E \setminus (A_1 \cup \cdots A_n)$. Since $|Y| > |X|$, we must have $|Y \cap A^*| > |X \cap A^*|$, or $|Y \cap A_i| > |X \cap A_i|$ for some i. In case $|Y \cap A^*| > |X \cap A^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. So we only need to study the case that $|Y \cap A_i| > |X \cap A_i|$ for some i. Without lost of generality we may assume $|Y \cap A_1| > |X \cap A_1|$.

Let $B_1, \ldots, B_m$ be the maximal sets in A_1 and let $B^* = A_1 \backslash (B_1 \cup \ldots B_m)$. Since $|Y \cap A_1| > |X \cap A_1|$, we have $|Y \cap B^*| > |X \cap B^*|$ or $|Y \cap B_i| > |X \cap B_i|$ for some i. Again if $|Y \cap B^*| > |X \cap B^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. Otherwise we can repeat this process for B_i satisfying $|Y \cap B_i| > |X \cap B_i|$. Since the ground set E is finite, we can find the required e in finite number of steps, and we are done.

4. Let $(j_1, \ldots, j_k)$ be a sequence of jobs that can be completed on time in that order. That is, $d_{j_i} \geq i$ for all i. Suppose that two adjacent jobs from the sequence are not in the order of their deadlines, that is, $d_{j_i} > d_{j_{i+1}}$. Then, they can be swapped without breaking feasibility: $d_{j_i} > d_{j_{i+1}} \geq i + 1$. Thus, we can sort the list according to increasing deadlines, while staying feasible.

   We will now prove that M is a matroid. Clearly $X \subseteq Y \in \mathcal{I} \implies X \in \mathcal{I}$. For (I_2), let $X, Y \in \mathcal{I}$, $|Y| > |X|$. Let $e \in Y \setminus X$ be the job having the latest deadline. Then $d_e \geq |Y| > |X|$ and it can be added to X without breaking feasibility.

   To find an optimal ordering of the jobs, we can assume $c_j > 0$ for all j and use the greedy algorithm.

5. For any $X \subseteq E$, let $\mathcal{F}_X = \{A \cap X : A \in \mathcal{F}\}$; if we assume wlog that $E \in \mathcal{F}$ then X is also in $\mathcal{F}_X$. We can see $\mathcal{F}_X$ is also a laminar family of sets. Call a set $S \in \mathcal{F}_X$, X-maximal in $T \subseteq E$ if $S \subset T$ and S is not contained in any other set in $\mathcal{F}_X$ that is properly contained in T. For each $T \in \mathcal{F}_X$, let $T_1, \ldots, T_{n_T}$ be X-maximal sets in T and let $T^* = T \backslash (T_1 \cup \ldots \cup T_{n_T})$. Then the rank of T satisfies

$$r(T) = \min\{r(T_1) + \cdots + r(T_{n_T}) + |T^*|, min_{A \in \mathcal{F}: T \subseteq A} k(A)\}, \tag{1}$$

   and this recursive formula also applies to X.