Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 13
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Solutions to Problem Set 3

1. From the description of the matching polytope, we know that the maximum size of a matching in a bipartite graph G is given by:

$$\nu(G) = \max\{1^T x : Ax \leq 1, x \geq 0\},$$

where 1 denotes a vector of all 1's and A is the vertex-edge incidence matrix (i.e., it has a row for each vertex and a column for each edge, and a 1 in this entry if the edge is incident to the vertex). By LP duality we know that

$$\nu(G) = \min\{1^T y : A^T y \geq 1, y \geq 0\},$$

where y has a component y_v for each vertex v of V. Since A is TUM (see Theorem 3.13) so is A^T. Thus this dual also has an optimum solution y that is integral. Furthermore, given the constraints, it is useless to have $y_v > 1$ and thus we can assume that for each $v \in V$, we have that $y_v \in \{0, 1\}$. We just need to observe not that $A^T y \geq 1$ is identical to saying that C is a vertex cover where $C = \{v \in V : y_v = 1\}$, and that $1^T y = |C|$. This proves König's theorem.

2. Let $I_i = [t_i, u_i) \subseteq \mathbb{R}$ be the time interval of activity i.

   (a) We could write the integer program:

$$\text{Max} \quad \sum_i p_i x_i$$

subject to:

$$\begin{aligned} x_i + x_j &\leq 1 \qquad && \forall i, j : I_i \cap I_j \neq \emptyset \\ x_i &\in \{0, 1\} && \forall i. \end{aligned}$$

And this would be correct for this subquestion. However, if we relax the integrality constraints some of the extreme points may be fractional (consider for example 3 identical intervals).

A better formulation as an integer program is:

$$\text{Max} \quad \sum_i p_i x_i$$

subject to:

$$\begin{aligned} \sum_{j : t_i \in I_j} x_j &\leq 1 \qquad && \forall i \\ x_i &\in \{0, 1\} && \forall i. \end{aligned}$$

This is a valid formulation of the problem as any feasible (integer) solution cannot have two intervals, say I_k and I_l, that overlap; indeed the constraint for $i = k$ or for $i = l$ would be violated.

(b) Sort the rows of A according to the left endpoint t_i that defines them. This shows that A has the "consecutive ones property" along columns, that is, every column is simply a (possible empty) group of zeros followed by a (possible empty) group of ones followed by a (possibly empty) group of zeros. Indeed, for the column that corresponds to x_j, we have a one in row i if $t_i \in I_j$, and this implies that the ones will be consecutive.

Now, we will show that any matrix with the consecutive ones property is totally unimodular. Consider any square submatrix T of A; it also has the consecutive ones property. For every row of T except the last, subtract the next row. This operation doesn't change the determinant, but leaves at most one 1 and one -1 in each column, while making the others 0. If there is a column with exactly one nonzero entry, then by expanding the determinant along that entry, we can consider a smaller sized matrix T. Otherwise, every column has exactly one 1 and one -1. If we pre-multiply such a matrix by the all ones vector, we get the 0 vector. That is, the matrix is singular and the determinant is 0. This shows that in all cases the determinant of T is 0, 1 or -1.

3. First consider the situation in which M_1 and M_2 are such that $M_1 \Delta M_2$ have more than one connected component. Consider one of these connected components, say $S \subseteq V$, and partition M_1 and M_2 into $M_1 = M_{1s} \cup M_{1t}$ and $M_2 = M_{2s} \cup M_{2t}$ where M_{1s} and M_{2s} correspond to the edges within S. By definition $M_{1s} \cup M_{2s} \neq \emptyset$. Now define two other matchings by $M_3 = M_{1s} \cup M_{2t}$ and $M_4 = M_{2s} \cup M_{1t}$. Observe that

$$\chi(M_1) + \chi(M_2) = \chi(M_3) + \chi(M_4)$$

which implies that any face that contains M_1 and M_2 will also contain M_3 and M_4, and thus cannot be an edge.

Conversely, suppose that $M_1 \Delta M_2$ has only one connected component, and say that this component has k_1 edges from M_1 and k_2 edges from M_2. We must have that $|k_1 - k_2| \leq 1$. Now consider the following cost function:

$$c_e = \begin{cases} 1 & e \in M_1 \cap M_2 \\ -1 & e \notin (M_1 \cup M_2) \\ k_2 & e \in M_1 \setminus M_2 \\ k_1 & e \in M_2 \setminus M_1. \end{cases}$$

Notice that $c(M_1) = c(M_2) = b$ where $b := |M_1 \cap M_2| + 2k_1k_2$ and for any other matching M we have that $c(M) < b$. Thus the valid inequality $c^T x \leq b$ induces a face with only teh incidence vectors of M_1 and M_2 has vertices. Thus the line segment between M_1 and M_2 defines an edge.

4. (a) Yes, the matrix is still TUM; we use Theorem 3.14 to prove it. Indeed, consider any subset R of its rows. Either the last row of the same matrix is not among

those selected in R and then the result follows from Theorem 3.13. On the other hand, if the last row is in R, then put it in R_2 and put all the other rows in R_1. The sum of the rows in R_1 will be a vector with all components $0, 1$, or 2, and once we substract the last row (with only 1's), we get a $\{0, +1, -1\}$-vector, proving that the matrix is TUM.

(b) This follows immediately from Theorem 3.12.

(c) Not true. Consider a complete bipartite graph with $|A| = |B| = 2$ and let E_1 be one of the perfect matchings and e_2 the other (both $|E_1|$ and $|E_2|$ equal 2). Take $k = 1$. The vector with $x_e = \frac{1}{2}$ for $e \in E_1 \cup E_2$ satisfies all the given inequalities, but is not in the convex hull of matchings with at most k edges from E_1.