Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 12
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Solutions to Problem Set 2

1. Let U be a minimizer set and M be a maximum matching. Note that each edge in M is either an edge of some $G[K_i]$ or it is adjacent to some vertex in U. Also note that the number of edges in M adjacent to U is at most $|U|$ and the number of edges in M from $G[K_i]$ is at most $\left\lfloor \frac{|K_i|}{2} \right\rfloor$. It follows that:

$$|M| = |[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| + \sum_{i=1}^{k} |M \cap E(G[K_k])|$$
$$\leq |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \frac{1}{2}(|V| + |U| - o(G \setminus U)) = |M|,$$

where the last equality holds because M is maximum and U is minimizer to the Tutte-Berge formula.

The previous formula implies that all the inequalities are equalities (if some of them is strict, we would have a contradiction). In particular we have that:

$$|[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| = |U|,$$
$$\text{and for every } i,\ |M \cap E(G[K_k])| = \left\lfloor \frac{|K_i|}{2} \right\rfloor.$$

In other words, M has exactly $|U|$ edges adjacent to some vertex in U, and for every i, M contains exacly $\left\lfloor \frac{|K_i|}{2} \right\rfloor$ edges from $G[K_i]$. In particular, $G[K_i]$ is perfectly matched for even components K_i and near-perfect matched for odd components.

From the previous analysis we know that every vertex u in U is matched to some vertex outside U. Since all the vertices of the even components are already matched, u must be matched to some vertex in an odd component K_i of $G \setminus U$.

Finally, since all the vertices in U are matched to some vertex outside U, and all the vertices in each even component are perfectly matched, we obtain that the only unmatched vertices must be in odd components of $G \setminus U$.

2. We will use the Tutte-Berge formula. Let $U \subseteq V$, and let W be any connected component of odd size of $G \setminus U$. Because G is 3-regular, there is an odd number of edges between W and U (this follows by just counting the edges incident to a vertex of W, and observing that the edges inside W will be counted twice). Moreover, those edges are a cutset. Thus, that cutset has at least 3 edges. Because this is happening for every connected component of odd size, the 3-regularity of G implies that $|U| \geq o(G \setminus U)$. In other words, by the Tutte-Berge formula, $\max_M |M| = |V|/2$.

3. We have:

$$Ax = b, x \geq 0 \text{ has no solution}$$
$$\text{iff}$$
$$Ax \leq b, -Ax \leq -b, -Ix \leq 0 \text{ has no solution}$$
$$\text{iff}$$
$$Cx \leq d \text{ has no solution where } C = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} \text{ and } d = \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$
$$\text{iff}$$
$$\exists (p, q, r) \geq 0 : A^T p - A^T q - Ir = 0, b^T p - b^T q < 0$$
$$\text{iff}$$
$$\exists (p, q) \geq 0 : A^T (p - q) \geq 0, b^T (p - q) < 0$$
$$\text{iff}$$
$$\exists y \geq 0 : A^T y \geq 0, b^T y < 0.$$

4. We know that every extreme point of a polytope is the unique solution for a system of equations obtained by selecting some of the inequalities defining the polytope and setting them as equalities. In particular, if x^* is a vertex of Q, then there exists a subset I of $\{1, \ldots, m\}$, where m is the number of rows of A, such that x^* is the unique solution of the system

$$\begin{aligned} (Ax)_i &= b_i, \text{ for all } i \in I \\ Cx &= d. \end{aligned}$$

But this implies that x^* is the unique solution for a system of equations obtained by selecting some of the inequalities that define P and setting them as equalities. Since $x^* \in P$, it follows that x^* is an extreme point of P. Therefore, all the vertices of Q are vertices of P.

5. Consider the family of permutations $\{\sigma_1, \ldots, \sigma_n\}$ given by:

$$\sigma_i : \begin{cases} \sigma_i(1) = i, \\ \sigma_i(i) = 1, \\ \sigma_i(j) = j, \quad j \notin \{1, i\}. \end{cases}$$

In other words, σ_1 is the identity permutation and σ_i is the permutation that transpose 1 and i. Let us prove that these n permutations are affinely independent. For that suppose that $\lambda_1, \ldots, \lambda_n$ are real numbers such that:

$$\sum_{i=1}^{n} \lambda_i \sigma_i = 0, \quad \sum_{i=1}^{n} \lambda_i = 0.$$

Then for every $j \neq 1$:

$$\begin{aligned}
0 = \sum_{i=1}^{n} \lambda_i \sigma_i(j) &= \left(\sum_{i \neq j} \lambda_i \cdot j \right) + \lambda_j \cdot 1 \\
&= \left(\sum_{i=1}^{n} \lambda_i \cdot j \right) - \lambda_j \cdot j + \lambda_j \cdot 1 \\
&= \underbrace{\left(\sum_{i=1}^{n} \lambda_i \right)}_{=0} \cdot j + \lambda_j \cdot (1 - j).
\end{aligned}$$

Hence, for every $j \neq 1$, $\lambda_j(1 - j) = 0$, which implies that $\lambda_j = 0$. Using that $\sum_{i=1}^{n} \lambda_i = 0$ we deduce that λ_1 is also 0. Therefore, the family $\{\sigma_1, \ldots, \sigma_n\}$ is a set of n affinely independent permutations.