Problem set 6

This problem set is due in class on May 3rd, 2007.

1. At some point during baseball season, each of n teams of the American League has already played several games. Suppose team i has won w_i games so far, and $g_{ij} = g_{ji}$ is the number of games that teams i and j have yet to play. No game ends in a tie, so each game gives one point to either team and 0 to the other. You would like to decide if your favorite team (Red Sox?), say team n, can still win. In other words, you would like to determine whether there exists an outcome to the games to be played (remember, with no ties) such that team n has at least as many victories as all the other teams (we allow team n to be tied for first place with other teams).

   Show that this problem can be solved as a maximum flow problem. Explain.

2. Consider the following orientation problem (as in the quiz). We are given an undirected graph $G = (V, E)$ and integer values $p(v)$ for every vertex $v \in V$. we would like to know if we can orient the edges of G such that the directed graph we obtain has at most $p(v)$ arcs incoming to v (the "indegree requirements"). In other words, for each edge $\{u, v\}$, we have to decide whether to orient it as (u, v) or as (v, u), and we would like at most $p(v)$ arcs oriented towards v.

   (a) Show that the problem can be formulated as a maximum flow problem. That is, show how to create a maximum flow problem such that, from its solution, you can decide whether or not the graph can be oriented and if so, it also gives the orientation.

   (b) Consider the case that the graph cannot be oriented and meet the indegree requirements. Prove from the max-flow min-cut theorem that there must exist a set $S \subseteq V$ such that $|E(S)| > \sum_{v \in S} p(v)$, where as usual $E(S)$ denotes the set of edges with both endpoints within S.

3. Suppose you are given an $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ with row sums $r_1, \cdots, r_m \in \mathbb{Z}$ and column sums $c_1, \cdots, c_n \in \mathbb{Z}$. Some of the entries might not be integral but the row sums and column sums are. Show that there exists a rounded matrix A' with the following properties:

   - row sums and column sums of A and A' are identical,
   - $a'_{ij} = \lceil a_{ij} \rceil$ or $a'_{ij} = \lfloor a_{ij} \rfloor$ (i.e. a'_{ij} is a_{ij} either rounded up or down.).

   (Hint. Think of flows.)

   By the way, this rounding is useful to the census bureau as they do not want to publish statistics that would give too much information on specific individuals. They want to be able to modify the entries without modifying row and column sums.

4. In lecture, we have defined flows on a directed graph. Suppose now we have an undirected graph $G = (V, E)$ with two designated vertices s and t, and assume that for every edge, we have an upper bound $u(e)$ (but for simplicity let's assume the lower bound $l(e) = 0$). A flow x will still be directed, but we will allow either $x(u, v) > 0$ or $x(v, u) > 0$ (but not both simultaneously) for the edge $e = (u, v)$ between u and v, i.e. the flow can only go from u to v or vice versa, but not both. The total flow going on e (in thus only one direction) is assumed to be at most the capacity $u(e)$.

   Show how to solve the undirected maximum flow problem by formulating it as a maximum flow problem on a directed graph, and explain why the two are equivalent.

5. Consider the $s - t$ flow problem on a directed graph in which every directed edge has a lower bound $l(e) = 0$ and an upper bound $u(e) = 1$.

   (a) Write the dual linear program of this maximum flow problem.

   (To make the dual a bit nicer to interpret, it is useful to add a variable $f \in \mathbb{R}$ in the primal, impose that the net flow of out of s equals f and that the net flow out of t equals $-f$, and maximize f. In the dual linear program, we will have one variable, say y_v, for every vertex $v \in V$ (which corresponds to either the flow balance constraint if $v \notin \{s, t\}$ or the newly introduced constraints if $v \in \{s, t\}$), and also one dual variable, say z_e for every directed edge (arc) $e \in E$ (corresponding to the inequalities $x_e \leq u(e)$).)

   (b) Does this dual always have an optimum solution that is integral?

   (c) Show that for any *integer* solution to the dual (i.e. $y_v, z_e \in \mathbb{Z}$) you can obtain an $s - t$ cut of the same value.

   (d) Now suppose you are given a non necessarily integral feasible solution for this dual of value V (in the dual linear program). Show how we can obtain a cut of value at most V.