Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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Handout 19
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Problem set 6: Solutions

1. (6 points). Let x_{ij} be the number of games that team i wins against j (excluding the already played games). x_{ij} is an integer, $0 \leq x_{ij} \leq g_{ij}$ and we have that for all $j \neq i$, $x_{ij} + x_{ji} = g_{ij}$. The total number of games won by team i is

$$w_i + \sum_{i \neq j} x_{ij}.$$

Team n can win iff it can win by winning all its remaining games. If team n wins all its remaining games, then in total it wins $T = w_n + \sum_{i<n} g_{in}$ games. This can happen if and only if there is a choice of x_{ij} where T is greater that the score of the other teams, that is, iff there is a feasible solution x for

$$\begin{aligned} T \geq w_i + \sum_{j=1, j \neq i}^{n} x_{ij} & \quad \forall i \leq n \\ 0 \leq x_{ij} \leq g_{ij} & \quad i \neq j \\ x_{ij} + x_{ji} = g_{ij} & \quad i < j. \end{aligned} \tag{1}$$

This is equivalent to the following flow problem: consider the graph having a source s, a sink t, a set of vertices A having one vertex per pair of teams among $\{1, \ldots, n-1\}$ (thus, $|A| = \binom{n-1}{2}$) and another set of vertices B having one vertex per team except n. The source s is connected to each vertex ij in A (with $i < j$) with capacity g_{ij} and every vertex i in B is connected to t with capacity $T - w_{ij}$. Every vertex ij in A is connected to i and j in B with capacity g_{ij}; this means that the flow along the arc from s to ij has to be divided among the two arcs from ij to i and j. If we can saturate the arc from s to ij then we can view x_{ij} $(i < j)$ as the flow along the arc from ij to i and x_{ji} as the flow from ij to j. Then, in view of Equation (1), team n can win iff there is a feasible flow saturating the edges between s and A. We also know that if there exists a flow of a certain value then there exists an *integral* flow of value at least as high since all capacities are integral. Thus, team n can win iff there is an integral flow saturating the edges between s and A.

2. (6 points).

   (a) Consider the graph G' having a source s, a sink t, a set of nodes A having one node per edge $\{u, v\} \in E$ of the undirected graph, and a set of vertices B having one node per vertex in the undirected graph. The source s is connected to each vertex $\{u, v\}$ in A with capacity 1, and every vertex v in B is connected to the sink t with capacity $p(v)$. Every vertex $\{i, j\}$ in A is connected to i and j in B with infinite capacity.

Note that for every orientation of the graph we can create a flow x in the network, by saturating all the arcs going out from s; setting, for every edge $\{u, v\}$ in the graph, a flow of 1 on the arc $(\{u, v\}, v)$ and 0 on the arc $(\{u, v\}, u)$ if $\{u, v\}$ is oriented as (u, v), or a flow of 1 on the arc $(\{u, v\}, u)$ and 0 on $(\{u, v\}, v)$ if the edge is oriented as (v, u); and assigning for every v a flow equal to the indegree of v to the arc (v, t).

Conversely, if there is an integral maximum flow that saturates the arcs going out from s, then we can obtain a valid orientation for G out of it by directing the edge $\{u, v\}$ towards u if the flow saturates $(\{u, v\}, u)$ or towards v otherwise.

We also know that if there exists a maximum flow that saturates all the arcs going out from s (i.e. a flow of value $|E|$), then there exists an *integral* flow of the same value since all capacities are integral (note that since the inflow in every node of A is 1, we can actually replace all the infinite capacities with any integer positive constant). Thus, G is orientable if and only if there is an maximum flow of value $|E|$ (that we can assume integral) and we can obtain the orientation by the method just described.

(b) If G cannot be oriented then the maximum flow for the previous network has flow value f strictly less than $|E|$. It follows by max-flow min-cut theorem that there is a (minimum) s-t cut C of value f. Note that $C = \{s\} \cup (A \cap C) \cup (B \cap C)$ is such that there is no arc from $A \cap C$ to $B \setminus C$, since those arcs have infinite capacity. Let S be the vertices in the original graph associated to $B \cap C$. It follows that the vertices in $A \cap C$ correspond to a set of edges in G that have both endpoints in S, i.e. to a subset of $E(S)$.

Using the previous observation, and that we can write the total capacity of C as the number of edges between s and $A \setminus C$, which is at least $|E \setminus E(S)|$, plus the sum of the capacities of the arcs from $A \cap C$ to V, we have:

$$|E| > f \geq |E| - |E(S)| + \sum_{v \in S} p(v).$$

Therefore $|E(S)| > \sum_{v \in S} p(v)$.

3. (6 points). Without loss of generality we can assume that $a_{ij} \in [0, 1)$ for all i, j: consider the matrix having only the fractional part of the entries of A, $a_{ij} - \lfloor a_{ij} \rfloor$.

Consider the following maximum st-flow problem: the digraph has as vertex set $\{s, t\} \cup R \cup C$, where R, C are sets of vertices having one vertex per row and column of A, respectively. There are arcs from s to every vertex $i \in R$ with capacity $\sum_{j=1}^{n} a_{ij}$ and there are arcs from every vertex $j \in C$ to t with capacity $\sum_{i=1}^{m} a_{ij}$. Finally, there is an arc from $i \in R$ to $j \in C$ with capacity 1 iff $a_{ij} > 0$.

There is a flow vector with flow a_{ij} on the arc from i to j, and it is a maximum flow as it saturates every arc incident to s. But, we know that there exists an optimum flow that is integral, as all the capacities are integral. Such an integer flow will put flow

$f_{ij} \in \{0,1\}$ in the edge between $i \in R$ and $j \in C$. The matrix having entries f_{ij} is the desired rounding of A.

4. (6 points). Define $G' = (V, A)$ to be the directed network with same vertex set as G and that contains for every edge $\{i, j\}$ in E, two arcs (i, j) and (j, i), both with capacity equal to $u_{\{i,j\}}$.

   Note that each s-t flow x in the undirected graph is, by definition, also a flow (with the same value) in the directed version. On the other hand if y is a flow in the directed graph, then we can define the flow z to be such that for every arc (i, j), $z_{ij} = \max\{0, y_{ij} - y_{ji}\}$. (In other words, if $y_{ij} \geq y_{ji}$ then we set z_{ij} to be equal to $y_{ij} - y_{ji}$ and z_{ji} to be 0, otherwise we set z_{ji} to be equal to $y_{ji} - y_{ij}$ and z_{ij} to be 0).

   Clearly z_{uv} satisfies the capacity constraints for every arc, and it also satisfies flow conservation. Thus z is a flow in the directed graph. Also, since for every edge $\{i, j\}$ of G at most one arc (i, j) or (j, i) carry flow, z is also a flow in the undirected graph, and it has the same flow value as y.

   We obtain that for every flow in the directed graph we can construct a flow in the undirected graph that has the same value and viceversa. Therefore, for every maximum flow in the directed case we can construct a maximum flow in the undirected case (and viceversa).

5. (6 points).

   (a) The flow problem is:

$$
\begin{aligned}
& \max f \\
\text{s.t.} \quad & \sum_{e \in \delta^+_G(u)} x_e - \sum_{e \in \delta^-_G(u)} x_e = 0 \quad \forall u \in V(G) \setminus \{s,t\} \\
& \sum_{e \in \delta^+_G(s)} x_e - \sum_{e \in \delta^-_G(s)} x_e - f = 0 \\
& \sum_{e \in \delta^+_G(t)} x_e - \sum_{e \in \delta^-_G(t)} x_e + f = 0 \\
& 0 \leq x_e \leq 1 \quad \forall e \in E(G).
\end{aligned}
$$

   The dual is:

$$
\begin{aligned}
& \min \sum_{e \in E(G)} z_e \\
\text{s.t.} \quad & y_{t(e)} - y_{h(e)} + z_e \geq 0 \quad \forall e \in E(G) \\
& -y_s + y_t = 1 \\
& z \geq 0,
\end{aligned}
$$

   where z_e is the dual variable corresponding to $x_e \leq 1$, and edge $e = (t(e), h(e))$.

(b) The constraint matrix of the dual is the transpose of the constraint matrix of the flow problem, which is totally unimodular. Thus, the dual always has an optimum solution that is integral.

(c) Let y, z be an integral optimal solution. Without loss of generality $y_s = 0$ (by adding $-y_s$ to every component of y). Consider the cut (S, T) given by $S = \{v \in V(G) : y_v = 0\}$, $T = V(G) \setminus S$. We can assume that $z_e = \max(0, y_{t(e)} - y_{h(e)})$ (as we are minimizing) and therefore we have that z_e is 1 if e belongs to the cut (S, T) and 0 otherwise. Thus we can find a cut of the same value as the dual solution.

(d) Let y, z be a feasible solution of value V. Label the vertices $v_1, \ldots, v_n$ in order of increasing value of y. We can choose $v_1 = s$, $v_n = t$. Consider the cuts $S_k = \{v_1, \ldots, v_k\}$, $k = 1, \ldots, n-1$. Let $\lambda_k = y_{v_{k+1}} - y_{v_k}$. We have that $\lambda_k \geq 0$ and $\sum \lambda_k = 1$. Also,

$$\sum_{k=1}^{n-1} \lambda_k |\delta(S_k)| \leq \sum_{e \in E(G)} \max\{y_{h(e)} - y_{t(e)}, 0\} \leq \sum_{e \in E(G)} z_e.$$

This implies that $\min_k |\delta(S_k)| \leq \sum_{e \in E(G)} z_e = V$, that is, one of the cuts S_k has value at most V.