Problem set 5: Solutions

1. (6 points).

   Let $G = (V, E)$ be a bipartite graph with partition A, B. Given a matching M, we know that any vertex cover C must contain at least one endpoint of each edge of C, therefore $|M| \leq |C|$, and so $\max_{M\colon \text{matching}} |M| \leq \min_{C\colon \text{vertex cover}} |C|$. We will use the min-max relation of matroid intersection to prove the other inequality.

   Consider the matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$, where

$$\mathcal{I}_1 = \{F \subseteq E : |F \cap \delta(v)| \leq 1, \text{ for all } v \in A\},$$
$$\mathcal{I}_2 = \{F \subseteq E : |F \cap \delta(v)| \leq 1, \text{ for all } v \in B\}.$$

   By the min-max relation for matroid intersection we have:

$$\max_{F \in \mathcal{I}_1 \cap \mathcal{I}_2} |F| = \min_{E_1 \subseteq E} r_1(E_1) + r_2(E \setminus E_1).$$

   Note that $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ if and only if F is the edge set of a matching. Therefore the LHS of the previous equation is the size of a maximum matching. Remember also that $r_1(E_1)$ equals the cardinality of the largest independent subset of E_1 with respect to $\mathcal{M}_1$ and thus equal the number of vertices of A covered by E_1 (and similarly for r_2).

   For any $E_1 \subseteq E$, define $\varphi(E_1)$ to be the following set of vertices:

$$\varphi(E_1) = \left(A \cap \bigcup_{e \in E_1} e\right) \cup \left(B \cap \bigcup_{e \in E \setminus E_1} e\right),$$

   where e should be understood as a set of its two endpoints. Then, by definition $\varphi(E_1) \cap A$ covers all the edges in E_1 and $\varphi(E_1) \cap B$ covers all the edges in $E \setminus E_1$. Therefore, $\varphi(E_1)$ is a vertex cover. Also, note that

$$|\varphi(E_1)| = \left|A \cap \bigcup_{e \in E_1} e\right| + \left|B \cap \bigcup_{e \in E \setminus E_1} e\right| = r_1(E_1) + r_2(E \setminus E_1).$$

   Hence, for every set E_1, $r_1(E_1) + r_2(E \setminus E_1)$ is at least the size of the minimum vertex cover. Therefore

$$\max_{M\colon \text{matching}} |M| = \max_{F \in \mathcal{I}_1 \cap \mathcal{I}_2} |F| = \min_{E_1 \subseteq E} r_1(E_1) + r_2(E \setminus E_1) \geq \min_{C\colon \text{vertex cover}} |C|.$$

2. (9 points). Define $\vec{E} = \{(u,v) \in V \times V : \{u,v\} \in E\}$ (observe that $|\vec{E}| = 2|E|$) and consider the matroids $\mathcal{M}_1 = (\vec{E}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\vec{E}, \mathcal{I}_2)$, where

$$\mathcal{I}_1 = \{F \subseteq \vec{E} : |F \cap \{(u,v),(v,u)\}| \leq 1, \text{ for all } \{u,v\} \in E\}.$$
$$\mathcal{I}_2 = \{F \subseteq \vec{E} : |F \cap \delta^-(v)| \leq k(v), \text{ for all } v \in V\}.$$

There exists an orientation satisfying the indegree requirements if and only if the largest independent set common to both matroids has cardinality $|E|$. By the min-max relation for matroid intersection we have:

$$\max_{F \in \mathcal{I}_1 \cap \mathcal{I}_2} |F| = \min_{E_1 \subseteq \vec{E}} r_1(E_1) + r_2(\vec{E} \setminus E_1).$$

Among all possible E_1 that minimizes $r_1(E_1) + r_2(\vec{E} \setminus E_1)$ choose one with maximum cardinality. Assume that there exists some $(u,v) \in E_1$ for which $(v,u) \notin E_1$. Then $E_1' = E_1 \cup \{(v,u)\}$ is a set of greater cardinality than E_1 such that

$$r_1(E_1') = r_1(E_1), \text{ and } r_2(\vec{E} \setminus E_1') \leq r_2(\vec{E} \setminus E_1),$$

which contradicts the choice of E_1. Therefore, for every $\{u,v\} \in E$, E_1 contains both (u,v) and (v,u) or none of them. From here we have that

$$r_1(E_1) = \frac{|E_1|}{2}.$$

Now assume that for some $v \in V$, we have $1 \leq |\delta^-(v) \cap (\vec{E} \setminus E_1)| \leq k(v)$. Then $E_1' = E_1 \cup (\delta^-(v) \cap (\vec{E} \setminus E_1))$ is a set of bigger cardinality than E_1 such that

$$\begin{aligned} r_1(E_1') &= r_1(E_1) + |\delta^-(v) \cap (\vec{E} \setminus E_1)|, \text{ and} \\ r_2(\vec{E} \setminus E_1') &= \sum_{u \in V} \min(k(u), |\delta^-(u) \cap (\vec{E} \setminus E_1')|) \leq \sum_{u \in V \setminus \{v\}} \min(k(u), |\delta^-(u) \cap (\vec{E} \setminus E_1)|) \\ &= r_2(E_1) - |\delta^-(v) \cap (\vec{E} \setminus E_1)|. \end{aligned}$$

This contradicts the choice of E_1. Therefore, for every v, $|\delta^-(v) \cap (\vec{E} \setminus E_1)|$ is either 0 or is greater than $k(v)$.

Now define

$$S = \{v \in V : |\delta^-(v) \cap (\vec{E} \setminus E_1)| > k(v)\}.$$

Then:

$$r_2(\vec{E} \setminus E_1) = \sum_{v \in V} \min(k(v), |\delta^-(v) \cap (\vec{E} \setminus E_1')|) = \sum_{v \in S} k(v).$$

Observe that if $u \notin S$, then $|\delta^-(u) \cap (\vec{E} \setminus E_1)| = 0$. Therefore all the arcs incident to u belongs to E_1.

Assume that there exists some $(u, v) \in E_1$ such that $u, v \in S$. Then we can define E_1' to be $E_1 \setminus \{(u, v), (v, u)\}$. But then $r_1(E_1') < r_1(E_1)$ while $r_2(\vec{E} \setminus E_1') = r_2(\vec{E} \setminus E_1)$. This contradicts the minimality of E_1. Using the previous observation, we conclude that:

$$E_1 = \vec{E} \setminus \vec{E}(S),$$

where $\vec{E}(S)$ is the set of arcs with both endpoints in S. It follows that:

$$r_1(E) = \frac{|\vec{E}| - |\vec{E}(S)|}{2} = |E| - |E(S)|, \text{ and } r_2(\vec{E} \setminus E) = \sum_{u \in S} k(u).$$

Finally, we know that an orientation exists if and only if $|E| \leq \max_{F \in \mathcal{I}_1 \cap \mathcal{I}_2} |F|$. By the min-max relation this happens if and only if

$$|E| \leq \min_{E_1} r_1(E) + r_2(\vec{E} \setminus E_1) = \min_{S \subseteq V} |E| - |E(S)| + \sum_{u \in S} k(u),$$

which is equivalent to the condition that for every $S \subseteq V$, $\sum_{u \in S} k(u) \geq |E(S)|$.