Problem set 4

This problem set is due in class on Thursday April 5th, 2007.

1. Consider the linear matroid (over the reals) defined by the 3×5 matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & -1 \\ 1 & 2 & 0 & 1 & -1 \end{pmatrix}.$$

   The ground set $E = \{1, 2, 3, 4, 5\}$ has cardinality 5, corresponds to the columns of A, and the independent sets are the set of columns which are linearly independent (over the reals).

   - Give all bases of this matroid.
   - Give all circuits of this matroid.
   - Take 2 circuits from your list with a non-empty intersection, and check that property $C3$ (page 52) is indeed satisfied for them.

2. Given a (not necessarily bipartite) graph $G = (V, E)$, let $\mathcal{I} = \{S \subseteq V :$ there exists a matching covering S. Show that $M = (V, \mathcal{I})$ is a matroid.

3. We are given n jobs that each take one unit of processing time. All jobs are available at time 0, and job j has a profit of c_j and a deadline d_j. The profit for job j will only be earned if the job completes by time d_j. The problem is to find an ordering of the jobs that maximizes the total profit. First, prove that if a subset of the jobs can be completed on time, then they can also be completed on time if they are scheduled in the order of their deadlines. Now, let $E(M) = \{1, 2, \cdots, n\}$ and let $\mathcal{I}(M) = \{J \subseteq E(M) : J$ can be completed on time $\}$. Prove that M is a matroid and describe how to find an optimal ordering for the jobs.

4. Given a family $A_1, A_2, \cdots, A_n$ of sets (they are not necessarily disjoint), a *transversal* is a set T such that $T = \{a_1, a_2, \cdots, a_n\}$, the a_i's are distinct, and $a_i \in A_i$ for all i. A partial transversal is a transversal for $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ for some subfamily of the A_i's.

   Show that the family of all partial transversals forms a matroid (on the ground set $E = \cup A_i$).

   Hint: Think of bipartite matchings.

5. Consider a graph $G = (V, E)$. Let $E(M) = E$ and $\mathcal{I}(M) = \{F_1 \cup F_2 : F_1, F_2$ are forests in $G\}$.

(a) Show that M is a matroid by showing that property $I3$ is satisfied ($I1$ and $I2$ are trivially satisfied).

(b) This implies that the greedy algorithm can find the maximum weight set of edges which can be partitioned into two forests. However, implementing the greedy algorithm is not at all easy (yet), as it is unclear how to test if a given set F of edges is independent (i.e. can be partitioned into 2 forests). [If you have time to kill (!) or are curious, try the analogous question for linear matroids: given a matrix A, when can the columns be partitioned into two sets of linearly independent columns?]

Give an example which shows that the following algorithm does *not* find a maximumw weight collection of edges which can be partitioned into 2 forests: find a maximum weight forest F in G, delete the edges of F, and find again a maximum weight forest in the remaining graph.

(c) (Optional for the undergrads) If we consider the union of k forests as the independent sets, would the resulting independence system still be a matroid?

6. Given an undirected graph $G = (V, E)$ and given integers $k(v)$ for every $v \in V$, show that the problem of deciding whether the graph can be oriented (i.e. replacing an edge (i, j) either by the arc (i, j) or by the arc (j, i)) into a directed graph $H = (V, A)$ so that, for every vertex v, the indegree $d^-(v)$ of v (the number of incoming edges to v) is at most $k(v)$ can be formulated as a matroid intersection problem.

   Hint: The ground set of both matroids should have cardinality $2|E|$.

7. (optional for the undergrads) Let $M = (E, \mathcal{I})$ be a matroid and let P be the corresponding matroid polytope, i.e. the convex hull of characteristic vectors of independent sets. Show that two independent sets I_1 and I_2 are adjacent on P if and only if either (i) $I_1 \subseteq I_2$ and $|I_1|+1 = |I_2|$, or (ii) $I_2 \subseteq I_1$ and $|I_2|+1 = |I_1|$, or (iii) $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$ and $I_1 \cup I_2 \notin \mathcal{I}$.

8. (optional for the undergrads) Recall the spanning tree game from lecture 1. It is a 2-player game played on a given graph $G = (V, E)$. Each player in turn selects an edge. Player 1 starts by deleting an edge, and then player 2 fixes an edge (which has not been deleted yet); an edge fixed cannot be deleted later on by the other player. Player 2 wins if he succeeds in constructing a spanning tree of the graph; otherwise, player 1 wins.

   In this problem, you need to describe a winning strategy for player 2 *if* the graph contains two edge-disjoint spanning trees. (This is actually an *if and only if* condition; if G does not have two edge-disjoint spanning trees, then player 1 has a winning strategy. You do not need to show the other direction.)

   HINT. Let F_1 and F_2 be the edges selected by player 1 and 2, respectively, in the first k rounds (so player 2 has just played). Let $E' = E \setminus (F_1 \cup F_2)$ be the remaining edges.

Try to maintain that there exists two disjoint subsets A and B in E' such that $F_2 \cup A$ and $F_2 \cup B$ are both spanning trees.