Massachusetts Institute of Technology
18.433: Combinatorial Optimization
Michel X. Goemans

Handout 14
April 9th, 2007
TA: José Soto

Problem set 4: Solutions

1. (5 points). The second and third rows of A are equal. This implies that the rank of A is at most 2 (actually it is 2 since A contains 2 l.i. column vectors). Therefore, $r(E) = 2$. Using this we have:

   - The set of bases of the matroid is formed by every pair of vectors in E that are linearly independent. In other wods, every pair of vectors (except the pair formed by the two first ones) define a basis of the matroid. Explicitly, the bases of the matroid are:

     $$B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{1,5\}, B_4 = \{2,3\}, B_5 = \{2,4\}, B_6 = \{2,5\},$$
     $$B_7 = \{3,4\}, B_8 = \{3,5\}, B_9 = \{4,5\}.$$

   - The circuits are the minimum dependent sets of the matroid. The only circuit of cardinality 2 is the one formed by the first two columns of A. All the other circuits have cardinality 3. Furthermore, any set of 3 element that does not contain $\{1,2\}$ is a circuit. Explicitly, the circuits of the matroid are:

     $$C_1 = \{1,2\}, C_2 = \{1,3,4\}, C_3 = \{1,3,5\}, C_4 = \{1,4,5\},$$
     $$C_5 = \{2,3,4\}, C_6 = \{2,3,5\}, C_7 = \{2,4,5\}, C_8 = \{3,4,5\}.$$

   - Consider, for example, circuits C_1 and C_2. Note that $C_1 \cap C_2 = \{1\}$. We must show that there exists a circuit in $C_1 \cup C_2 - 1 = \{2,3,4\}$. Clearly C_5 is that circuit.

2. (7 points). We only need to check that axioms $(I1)$,$(I2)$ and $(I3)$ are satisfied for M.

   $(I1)$ Note that no matter what graph G is, the empty set $\varnothing$ is always covered by the empty matching. Hence $\varnothing \in \mathcal{I}$.

   $(I2)$ Let $X \subset Y \in \mathcal{I}$. Since Y is an independent set, there exists a matching M_Y that covers Y. This matching also covers X. Hence X is an independent set.

   $(I3)$ Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$. It follows that there exist matchings M_X and M_Y such that M_X covers X and M_Y covers Y. Consider the graph $G' = (V, M_X \Delta M_Y)$. The set of edges of G' is the union of paths and cycles.

   If M_X covers some element y in $Y \setminus X$. Then $X + y$ is an independent set.

   Otherwise, all the vertices in $Y \setminus X$ are of degree 1 in G'. Since $|Y| > |X|$, we have $|Y \setminus X| > |X \setminus Y|$. Therefore, by the previous observation, there are more degree 1 vertices in $Y \setminus X$ than in $X \setminus Y$. It follows that there exists a path P in the decomposition of G' starting in a vertex $y \in Y \setminus X$ and not ending in X. We conclude that $M_X \Delta P$ is a matching of G that covers $X \cup \{y\}$. Thus, $X + y$ is an independent set.

3. (7 points). Let $(j_1, \ldots, j_k)$ be a sequence of jobs that can be completed on time in that order. That is, $d_{j_i} \geq i$ for all i. Suppose that two adjacent jobs from the sequence are not in the order of their deadlines, that is, $d_{j_i} > d_{j_{i+1}}$. Then, they can be swapped without breaking feasibility: $d_{j_i} > d_{j_{i+1}} \geq i+1$. Thus, we can sort the list according to increasing deadlines, while staying feasible.

   We will now prove that M is a matroid. Clearly $\emptyset \in \mathcal{I}$ and $X \subseteq Y \in \mathcal{I} \implies X \in \mathcal{I}$. For the third independence axiom, let $X, Y \in \mathcal{I}$, $|Y| > |X|$. Let $e \in Y \setminus X$ be the job having the latest deadline. Then $d_e \geq |Y| > |X|$ and it can be added to X without breaking feasibility.

   To find an optimal ordering of the jobs, we can assume $c_j > 0$ for all j and use the greedy algorithm.

4. (7 points). It is easy to see that the first 2 axioms are satisfied. For the third, consider the following bipartite graph: one set of vertices is $A = \cup A_i$ and the other is $B = \{A_i : 1 \leq i \leq n\}$. A pair of vertices $a \in A$ and $A_i \in B$ forms an edge iff $a \in A_i$. There is a correspondence between partial transversals of the family of sets and matchings of the graph: given a matching, the corresponding partial transversal is given by all the vertices in A that have an edge of the matching incident to them. Consider two transversals X, Y such that $|X| < |Y|$, with associated matchings with edges M and N. Consider the sets $Y \setminus X$ and $X \setminus Y$ and alternating paths in $M \triangle N$ starting at vertices in $Y \setminus X$. Because of the cardinality condition we have that $|Y \setminus X| > |X \setminus Y|$, which implies that not all such alternating paths can end at vertices in $X \setminus Y$, that is, there exists $a \in Y \setminus X$ such that the alternating path in $M \triangle N$ starting from it ends at a vertex in B. Then $X \cup \{a\}$ is also a partial transversal, as we can augment the matching M by means of the alternating path starting at a.

5. (7 points)

   (a) The main axiom to check is I3 (the other 2 are easy to check). Let $S, T \in \mathcal{I}(M)$ with $|T| > |S|$. Assume that $T = T_1 \cup T_2$ and $S = S_1 \cup S_2$, where T_1 and T_2 are *disjoint* forests, and so are S_1 and S_2. There are many ways of decomposing S and T into forests, but for now, let's take any decomposition. Now, either $|T_1| > |S_1|$ or $|T_2| > |S_2|$; by symmetry assume that $|T_1| > |S_1|$. The fact that the forests of a graph form a matroid means that there exists $e \in T_1 \setminus S_1$ such that $S_1 \cup \{e\}$ is a forest. Now there are 2 cases.

   If $e \notin S_2$, then we can replace S by $S \cup \{e\} = (S_1 \cup \{e\}) \cup S_2 \in \mathcal{I}(M)$ and we are done.

   If $e \in S_2$ then we just replace S_1 by $S_1' = S_1 \cup \{e\}$ and S_2 by $S_2' = S_2 \setminus \{e\}$. Notice that $S = S_1' \cup S_2'$. We can now repeat with this new decomposition. In so doing, we have increased $|S_1 \cap T_1| + |S_2 \cap T_2|$ and therefore, this second case cannot happen forever. (We should also say that $|S_1 \cap T_1| + |S_2 \cap T_2|$ also increases in the symmetric case when $|T_2| > |S_2|$ and we are moving an element $e \in T_2 \setminus S_2$ from S_1 to S_2.)

(b) Consider K_4 (the complete graph on 4 vertices) with all the edges having weight 1 and add a vertex connected to all the other 4 vertices with edges of weight 10. Then the given algorithm will firstly choose the forest given by the "star" of all the edges of weight 10, and then it will choose any tree within the remaining edges, giving a total weight of 43. But there are solutions that achieve a weight of 44: one can choose one forest to have only one edge of weight 10 and 3 edges of weight 1 and then as the second forest the rest of the edges of weight 10 plus one edge of weight 1.

(c) (optional, +4 points). Yes, this would still be a matroid. We can adapt the proof in (a). Assume that $S = S_1 \cup S_2 \cup \cdots S_k$ and that $T = T_1 \cup \cdots T_k$ and that $|T| > |S|$. Assume that the T_i's are disjoint, and so are the S_i's. Assume that the S_i and the T_i's are chosen so that $\sum_i |S_i \cap T_i|$ is maximized. One of the $|T_i|$'s must be larger than $|S_i|$, say $|T_1| > |S_1|$. Now we know that there must exist $e \in T_1 \setminus S_1$ such that $S_1 \cup \{e\}$ is a forest. If e was also in S_j, this would violate the maximality of $\sum_i |S_i \cap T_i|$ as we could move e from S_j to S_1. Thus $e \notin S_j$ for any j and therefore we can add e to S_1, and we get that $S \cup \{e\}$ is the union of k forests.

6. (7 points). Consider the ground set D given by:

$$D = \overleftrightarrow{E} = \{(v, w) \in V \times V : \{v, w\} \in E\}$$

i.e. D is the set of all $2|E|$ possible arcs in an orientation of E. For a given $e = \{v, w\}$ denote by $D_e = \{(v, w), (w, v)\}$ to the set of the two possible orientations of e. Also, for every $v \in V$ define $\delta^-(v) = \{(w, v) : \{w, v\} \in E\}$ to the set of arcs of D incident to v that are oriented towards v. It is easy to see that the families $\{D_e : e \in E\}$ and $\{\delta^-(v) : v \in V\}$ are both partitions of the set D.

Using that we can define two partition matroids M_1 and M_2 over the ground set D, where:

$$\mathcal{I}(M_1) = \{F \subseteq D : |F \cap D_e| \leq 1, \text{ for every } e \in E\}.$$
$$\mathcal{I}(M_2) = \{F \subseteq D : |F \cap \delta^-(v)| \leq k(v), \text{ for every } v \in V\}.$$

Note that every independent set in $\mathcal{I}(M_1)$ of size $|E|$ corresponds to an orientation of the graph G. Also, every independent set in $\mathcal{I}(M_2)$ corresponds to a digraph (with possible antiparallel edges) in which every vertex v has indegree at most $k(v)$.

Therefore, G admits an orientation in which every vertex has indegree at most $k(v)$ if and only if there exists an independent set of size $|E|$ in $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$.

7. (Optional, 8 points).

First, let us prove that the conditions are sufficient.

Consider two independent set I_1 and I_2 such that (i) holds. Let f be the only element in $I_2 \setminus I_1$, and consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 1, & \text{if } e \in I_1, \\ 0, & \text{if } e = f, \\ -1, & \text{if } e \notin I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2. Therefore I_1 and I_2 are adjacent. The case where (ii) holds is analogous.

Now, assume that I_1 and I_2 satisfy (iii). For this case let f be the only element in $I_2 \setminus I_1$ and g be the only element in $I_1 \setminus I_2$. Consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 2, & \text{if } e \in I_1 \cap I_2, \\ 1, & \text{if } e = f, \text{ or } e = g \\ -1, & \text{if } e \notin I_1 \cup I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2, and so they are adjacent in the matroid polytope.

Now let us prove that the conditions are necessary.

Assume that I_1 and I_2 are a pair of adjacent independent sets and let $c : E \to \mathbb{R}$ be a cost function that is maximized only by I_1 and I_2. In particular note that $c(e) \geq 0$ for every element in $I_1 \cup I_2$. Assume w.l.o.g. that $|I_1| \leq |I_2|$.

Case 1:$|I_2| > |I_1|$. By the exchange axiom ($I3$), there exists an element $f \in I_2 \setminus I_1$ such that $I_1 + f$ is an independent set and, by a previous observation, it has weight greater or equal than the weight of I_1. Since I_1 is optimum it follows that so is $I_1 + f$. Since I_2 and I_1 are the only optimums, it follows that $I_2 = I_1 + f$. Therefore, (i) holds.

Case 2: $|I_2| = |I_1|$. Let f be the element in $I_1 \Delta I_2 = I_1 \setminus I_2 \cup I_2 \setminus I_1$ with minimum cost. Assume w.l.o.g. that $f \in I_1$. Clearly, $I_1 - f$ is an independent set and $|I_1 - f| < |I_2|$. It follows that there exists an element $g \in I_2 \setminus I_1$ such that $I_1 - f + g$ is an independent set. By choice of f, $c(I_1 - f + g) = c(I_1) - c(f) + c(g) \geq c(I_1)$. But then $I_1 - f + g$ is also a maximum weight independent set. Since I_2 and I_1 were the only optimums, it follows that $I_2 = I_1 - f + g$, which implies that $|I_2 \setminus I_1| = |I_2 \setminus I_2| = 1$.

To conclude that (iii) holds, we only need to show that $I_1 \cup I_2 \notin \mathcal{I}$. But this is easy to see since, in other case, using that $c(e) \geq 0$ for every $e \in I_1 \cup I_2$, we would have that $c(I_1 \cup I_2) \geq c(I_1)$. This implies that $I_1 \cup I_2$ is another optimum (different from I_1 and I_2), which contradicts the adjacency condition of I_1 and I_2.

8. (Optional, 8 points). The strategy will maintain, in every round two disjoint sets A and B such that both A union the edges selected by player 2 and B union the edges selected by player 2 are spanning trees.

Let T_1 and T_2 be two edge-disjoint spanning trees of G. At the beginning of the game set $A = T_1$ and $B = T_2$, they satisfy our invariant. Let F_1 and F_2 be the sets of edges selected so far by both players respectively (at the beginning $F_1 = F_2 = \varnothing$).

Assume that at certain point of the game, it is player 1 turn and that he selects edge e. Player 2 needs to specify what edge to select and how to update A and B. The strategy is as follows:

Suppose $e \in A$. By the invariant, both $F_2 \cup A$ and $F_2 \cup B$ are spanning trees, i.e. they are independent sets in the graphic matroid. Since $|F_2 \cup B| > |F_2 \cup A - e|$ there exists an edge f in B such that $F_2 \cup A - e + f$ is an independent set, and by a cardinality argument, it is also a spanning tree. Let $A' = A - e$, $B' = B - f$ and $F_2' = F_2 + f$. Then $F_2' \cup A' = F_2 \cup A - e + f$ and $F_2' \cup B' = F_2 \cup B$ are both spanning trees. It follows that if player 2 chooses f and updates $A \leftarrow A'$ and $B \leftarrow B'$ the invariant will still hold.

Now suppose $e \in B$. By a symmetric argument, there is an $f \in A$ such that if player 2 chooses f and updates $A \leftarrow A - f$ and $B \leftarrow B - e$, the invariant will still hold.

The last case is when $e \notin A \cup B$. For this case, let f be any edge in A, and note that since $B \cup F_2$ is a spanning tree, $B \cup F_2 + f$ contrains a unique cycle. By letting g to be any edge different from f in that cycle, we conclude that $B \cup F_2 + f - g$ is also a spanning tree. Therefore, if player 2 chooses f and updates $A \leftarrow A - f$, and $B \leftarrow B - g$, the invariant will still hold.

Note that using this strategy, player 2 can always answer a move of player 1 while maintaining the invariant. It follows that after $n-1$ rounds, $F_2 \cup A$ and $F_2 \cup B$ will be both spanning trees. But since $|F_2| = n - 1$ we have that F_2 itself must be a spanning tree, which means player 2 won in the last move.