Problem set 2

This problem set is due in class on Tuesday March 6th, 2007

1. Let $G = (V, E)$ be a (not necessarily bipartite) graph. Given a set $S \subseteq V$, suppose that there exists a matching M covering S (i.e. S is a subset of the matched vertices in M). Prove that there exists a *maximum* matching M^* covering S as well.

2. Let U be any minimizer in the Tutte-Berge formula, i.e.

$$\max_M |M| = \frac{1}{2}(|V| + |U| - o(G \setminus U)).$$

   Let $K_1, \cdots, K_k$ be the connected components of $G \setminus U$. Show that, for *any* maximum matching M, we must have that

   (a) M contains exactly $\lfloor \frac{|K_i|}{2} \rfloor$ edges from $G[K_i]$ (the subgraph of G induced by the vertices in K_i), i.e. $G[K_i]$ is perfectly matched for the even components K_i and near-perfectly matched for the odd components.

   (b) Each vertex $u \in U$ is matched to a vertex v in an odd component K_i of $G \setminus U$.

   (c) the only unmatched vertices must be in odd components K_i of $G \setminus U$.

3. Could there be several minimizers U in the Tutte-Berge formula? Either give an example with several sets U achieving the minimum, or prove that the set U si unique.

4. (optional for the undergrads.) Given a graph $G = (V, E)$, an *inessential* vertex is a vertex v such that there exists a *maximum* matching of G not covering v. Let B be the set of all inessential vertices in G (e.g., if G has a perfect matching then $B = \emptyset$). Let C denote the set of vertices not in B but adjacent to at least one vertex in B (thus, if $B = \emptyset$ then $C = \emptyset$). Let $D = V \setminus (B \cup C)$. The triple $\{B, C, D\}$ is called the Edmonds-Gallai partition of G. Show that $U = C$ is a minimizer in the Tutte-Berge formula. (In particular, this means that in the Tutte-Berge formula we can assume that U is such that the union of the odd connected components of $G \setminus U$ is precisely the set of inessential vertices.)

5. An edge cover of a graph is a subset F of edges such that for every vertex $v \in V$ there exists an edge of F adjacent to it. Consider a (not necessarily bipartite) connected graph $G = (V, E)$ with $n = |V|$ vertices, and assume that the maximum matching has size $\nu(G)$. Prove that the minimum cardinality of an edge cover is precisely $n - \nu(G)$.

6. Show that any 3-regular 2-edge-connected graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. (A 2-edge-connected graph has at least 2 edges in every cutset; a cutset being the edges between S and $V \setminus S$ for some vertex set S.)