Problem set 2

This problem set is due in class on Tuesday March 6th, 2007

1. Let $G = (V, E)$ be a (not necessarily bipartite) graph. Given a set $S \subseteq V$, suppose that there exists a matching $M$ covering $S$ (i.e. $S$ is a subset of the matched vertices in $M$). Prove that there exists a maximum matching $M^*$ covering $S$ as well.

2. Let $U$ be any minimizer in the Tutte-Berge formula, i.e.

$$\max_M |M| = \frac{1}{2}(|V| + |U| - o(G \setminus U)).$$

Let $K_1, \ldots, K_k$ be the connected components of $G \setminus U$. Show that, for any maximum matching $M$, we must have that

(a) $M$ contains exactly $\left\lfloor \frac{|K_i|}{2} \right\rfloor$ edges from $G[K_i]$ (the subgraph of $G$ induced by the vertices in $K_i$), i.e. $G[K_i]$ is perfectly matched for the even components $K_i$ and near-perfectly matched for the odd components.

(b) Each vertex $u \in U$ is matched to a vertex $v$ in an odd component $K_i$ of $G \setminus U$.

(c) the only unmatched vertices must be in odd components $K_i$ of $G \setminus U$.

3. Could there be several minimizers $U$ in the Tutte-Berge formula? Either give an example with several sets $U$ achieving the minimum, or prove that the set $U$ is unique.

4. (optional for the undergrads.) Given a graph $G = (V, E)$, an inessential vertex is a vertex $v$ such that there exists a maximum matching of $G$ not covering $v$. Let $B$ be the set of all inessential vertices in $G$ (e.g., if $G$ has a perfect matching then $B = \emptyset$). Let $C$ denote the set of vertices not in $B$ but adjacent to at least one vertex in $B$ (thus, if $B = \emptyset$ then $C = \emptyset$). Let $D = V \setminus (B \cup C)$. The triple $\{B, C, D\}$ is called the Edmonds-Gallai partition of $G$. Show that $U = C$ is a minimizer in the Tutte-Berge formula. (In particular, this means that in the Tutte-Berge formula we can assume that $U$ is such that the union of the odd connected components of $G \setminus U$ is precisely the set of inessential vertices.)

5. An edge cover of a graph is a subset $F$ of edges such that for every vertex $v \in V$ there exists an edge of $F$ adjacent to it. Consider a (not necessarily bipartite) connected graph $G = (V, E)$ with $n = |V|$ vertices, and assume that the maximum matching has size $\nu(G)$. Prove that the minimum cardinality of an edge cover is precisely $n - \nu(G)$.

6. Show that any 3-regular 2-edge-connected graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. (A 2-edge-connected graph has at least 2 edges in every cutset; a cutset being the edges between $S$ and $V \setminus S$ for some vertex set $S$.)