Problem set 2: Solutions

1. (6 points). By hypothesis, there exist a matching M that covers S. Let M^* be one matching covering S with maximum cardinality, and assume that M^* is not a maximum matching. Therefore, there exists an M^*-augmenting path P. Note that if we augment over P, then none of the vertices that were previous covered by M^* get exposed. But then, $M^*\Delta P$ is a matching covering S with strictly more edges than M^* which is a contradiction. It follows that M^* is a maximum matching covering S.

2. (8 points). Let U be a minimizer set and M be a maximum matching. Note that each edge in M is either an edge of some $G[K_i]$ or it is adjacent to some vertex in U. Also note that the number of edges in M adjacent to U is at most $|U|$ and the number of edges in M from $G[K_i]$ is at most $\left\lfloor \frac{|K_i|}{2} \right\rfloor$. It follows that:

$$\begin{aligned} |M| &= |[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| + \sum_{i=1}^{k} |M \cap E(G[K_k])| \\ &\leq |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \frac{1}{2}(|V| + |U| - o(G \setminus U)) = |M|, \end{aligned}$$

where the last equality holds because M is maximum and U is minimizer to the Tutte-Berge formula.

The previous formula implies that all the inequalities are equalities (if some of them is strict, we would have a contradiction). In particular we have that:

$$\begin{aligned} |[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| &= |U|, \\ \text{and for every } i,\ |M \cap E(G[K_k])| &= \left\lfloor \frac{|K_i|}{2} \right\rfloor. \end{aligned}$$

In other words, M has exactly $|U|$ edges adjacent to some vertex in U, and for every i, M contains exacly $\left\lfloor \frac{|K_i|}{2} \right\rfloor$ edges from $G[K_i]$. In particular, $G[K_i]$ is perfectly matched for even components K_i and near-perfect matched for odd components.

From the previous analysis we know that every vertex u in U is matched to some vertex outside U. Since all the vertices of the even components are already matched, u must be matched to some vertex in an odd component K_i of $G \setminus U$.

Finally, since all the vertices in U are matched to some vertex outside U, and all the vertices in each even component are perfectly matched, we obtain that the only unmatched vertices must be in odd components of $G \setminus U$.

3. (4 points). There are many counterexamples. The simplest one is to consider the graph G with two vertices v_1 and v_2 joined by one edge. In this graph the maximum matching has cardinality 1. Consider $U_1 = \{v_1\}$, $U_2 = \{v_2\}$, $U_3 = \varnothing$. For all of them,

$$\frac{1}{2}\left(|U_i| + |V| - o(G \setminus U_i)\right) = 1.$$

Therefore any of them is a minimizer set.

4. (10 points). From exercise 2, we know that all inessential vertices will be in odd components of $G \setminus U$. The converse is not necessarily true; it depends on our choice of minimizer in the Tutte-Berge formula (from exercise 3, we know that there might be many such minimizers). There are several ways of proving the result. One way is to show that the minimizer U obtained in the proof of the Tutte-Berge formula in lecture has the right properties, i.e. it is such that any vertex v in one of the odd components K_i of $G \setminus U$ is inessential. This would show that the set of inessential vertices is precisely the union of odd components of $G \setminus U$ for the set U obtained in the lecture notes.

To prove this claim, consider any vertex v in any odd component of $G \setminus U$. Let v' be the vertex in the final shrunk graph $G' = (V', E')$ at the end of the algorithm. By construction, we know that v' is labelled *Even*. In this final graph G', v' is inessential. Indeed, the fact that v' is labelled *Even* means that there exists an even alternating path P for the final matching M from an exposed vertex to v'. Taking the symmetric difference of M and P gives a maximum matching in which v' is exposed. Now let us see what happens when we expand the blossoms. To get G', we shrank blossoms and $v = v_0$ successively became v_1, v_2, $\cdots, v_k = v'$; let $G = G_0, G_1, \cdots, G_k = G'$ be the corresponding graphs and let B_i be the blossom that got shrunk in G_i (to get G_{i+1}). We will show by (backward) induction that as expand the blossoms, G_i still has a maximum matching M_i leaving v_i exposed. Assume this is true for $i+1$. We can add to M_{i+1} $\lfloor |B_i|/2 \rfloor$ edges of B_i to get a matching M_i that leaves v_i exposed (irrespective of whether v_i belongs or not to the blossom). By induction, we therefore get a maximum matching M_0 leaving v exposed, and this shows that v is inessential.

Let B be the inessential vertices and consider the set C of neighbors of B not in B. As B is the union of the odd connected components of $G \setminus U$, we have that $C \subseteq U$. Also any odd connected component in $G \setminus U$ is also an odd connected component in $G \setminus C$ (there might be more in $G \setminus C$), showing that $o(G \setminus C) \geq o(G \setminus U)$. Thus, $|C| - o(G \setminus C) \leq |U| - o(G \setminus U)$, implying that (i) C is also a minimizer for the Tutte-Berge formula and (ii) U actually equals C. This completes the proof.

5. (6 points). Let $\rho(G)$ be the size of a minimum edge cover. A maximum matching covers $2\nu(G)$ vertices, and, because of the connectedness, the $n - 2\nu(G)$ remaining vertices can be covered by no more than $n - 2\nu(G)$ edges. This edges and the maximum matching are thus an edge cover of size $n - \nu(G)$. On the other hand, a minimum edge cover has

to be a forest (an acyclic graph). (Indeed, if it has any cycles then the removal of any edge of the cycle would still give an edge cover, of smaller cardinality.) The number of connected components of this forest is precisely $n - \rho(G)$ (because every component is a tree, and a tree on k vertices has $k - 1$ edges), and one can take one edge per component to get a matching. That is, $\nu(G) \geq n - \rho(G)$.

6. (6 points). We will use the Tutte-Berge formula. Let $U \subseteq V$, and let W be any connected component of odd size of $G \setminus U$. Because G is 3-regular, there is an odd number of edges between W and U (this follows by just counting the edges incident to a vertex of W, and observing that the edges inside W will be counted twice). Moreover, those edges are a cutset. Thus, that cutset has at least 3 edges. Because this is happening for every connected component of odd size, the 3-regularity of G implies that $|U| \geq o(G \setminus U)$. In other words, by the Tutte-Berge formula, $\max_M |M| = |V|/2$.