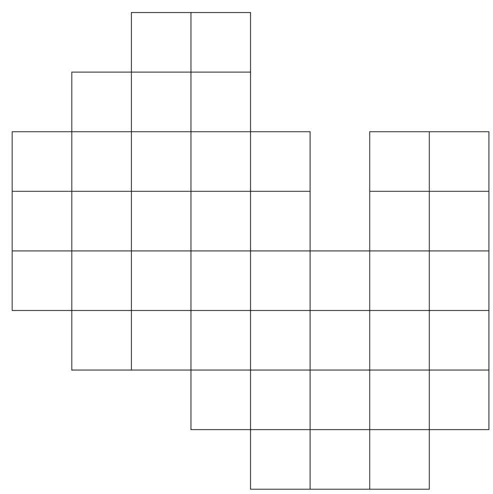
Problem set 1

This problem set is due in class on February 22nd, 2007.

1. Show that in any graph $G = (V, E)$ (not necessarily bipartite), the size of *any maximal* matching M (i.e. a matching M in which one cannot add an edge while keeping it a matching) is at least half the size of a *maximum* matching M^*.

2. Can the following figure be tiled by dominoes (a domino being 2 adjacent squares)? Give a tiling or a short proof that no tiling exists.



3. Consider a bipartite graph $G = (V, E)$ with bipartition (A, B): $V = A \cup B$. Assume that, for some vertex sets $A_1 \subseteq A$ and $B_1 \subseteq B$, there exists a matching M_A covering all vertices in A_1 and a matching M_B covering all vertices in B_1. Prove that there always exists a matching covering all vertices in $A_1 \cup B_1$.

4. Consider the following 2-person game on a (not necessarily bipartite) graph $G = (V, E)$. Players 1 and 2 alternate and each selects a (yet unchosen) edge e of the graph so that e together with the previously selected edges form a simple path. The first player unable to select such an edge loses. Show that if G has a *perfect* matching then player 1 has a winning strategy.

5. Given a bipartite graph $G = (V, E)$ with bipartition A, B ($V = A \cup B$), Hall's theorem gives a condition for the existence of a matching of size $|A|$ (i.e. every vertex in A is matched): For every $S \subseteq A$ we have $|N(S)| \geq |S|$, where $N(S) = \{b \in B : \exists a \in S$ with $(a, b) \in E\}$. Deduce Hall's theorem from König's theorem.

6. Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

   (a) by using König's theorem,

   (b) by using the linear programming formulation we saw in lecture for the assignment problem.

7. Using the previous problem, show that the edges of a k-regular bipartite graph G can be partitioned into k matchings (i.e. the number of colors needed to color the edges of a k-regular bipartite graph such that no two edges with an endpoint in common have the same color is precisely k).

   (Optional: Are this result and the one in the previous exercise also true also for non-bipartite graphs?)

8. In lecture, we have shown that, given a bipartite graph $G = (V, E)$, the following linear program always has an optimum $0 - 1$ solution:

$$\text{Min} \quad \sum_{i,j} c_{ij} x_{ij}$$

subject to:

$$\sum_j x_{ij} = 1 \qquad i \in A$$

$$\sum_i x_{ij} = 1 \qquad j \in B$$

$$x_{ij} \geq 0 \qquad i \in A, j \in B$$

   Prove this in the following way. Take a (possibly non-integral) *optimum* solution x^*. If there are many optimum solutions, take one with as few non-integral values x^*_{ij} as possible. Show that, if x^* is not integral, there exists a cycle C with all edges $e = (i, j) \in C$ having a non-integral value x^*_{ij}. Now show how to derive another optimum solution with fewer non-integral values, leading to a contradiction.