Massachusetts Institute of Technology
18.433: Combinatorial Optimization
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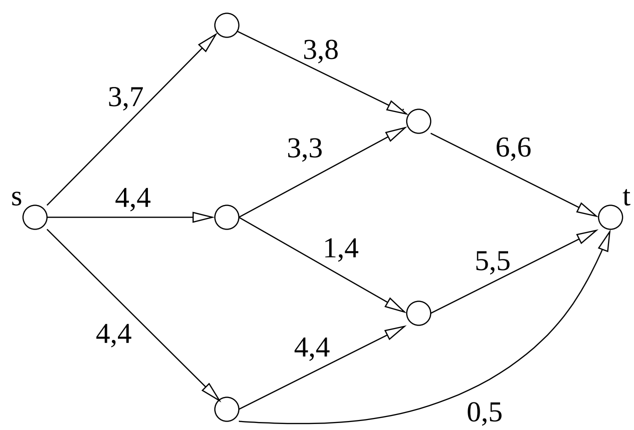
Handout 21
December 16th, 2005

FINAL

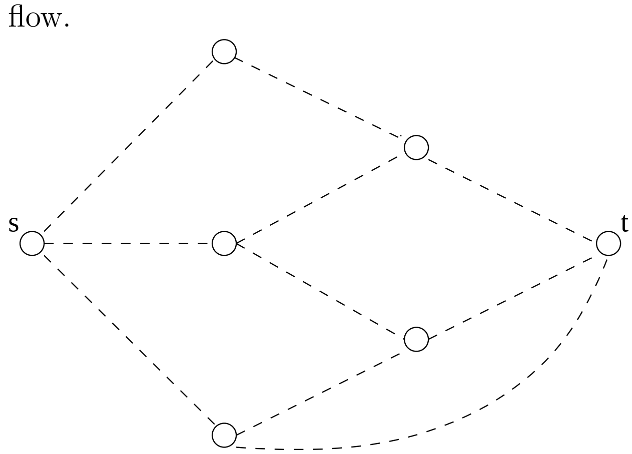
This exam is closed book. You are allowed to have two sheets of paper with notes. Happy Holidays!

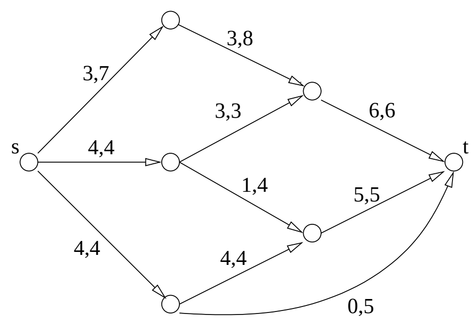
Your Name:

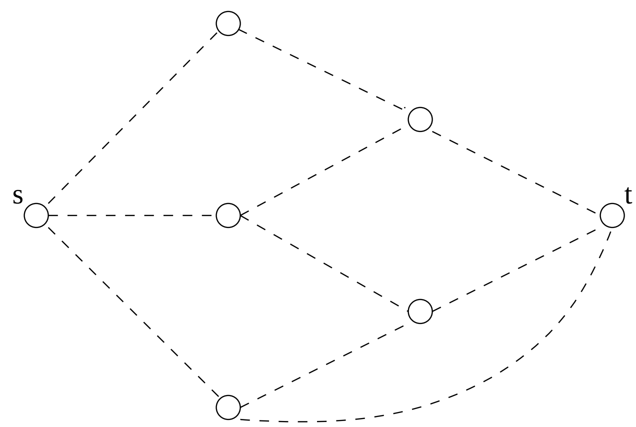
1. Consider the following flow from s to t in the directed graph G below. There are 2 numbers next to each arc e: the first is the flow x_e and the second is the capacity u_e (the lower capacities are all 0).



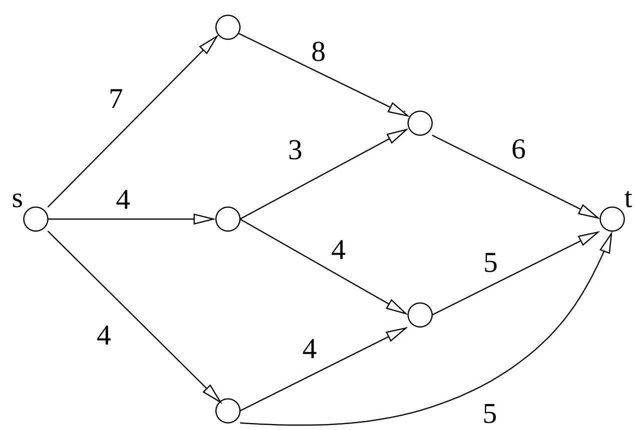
(a) Is the flow maximum? If yes, justify. If not, augment it and give a maximum flow.



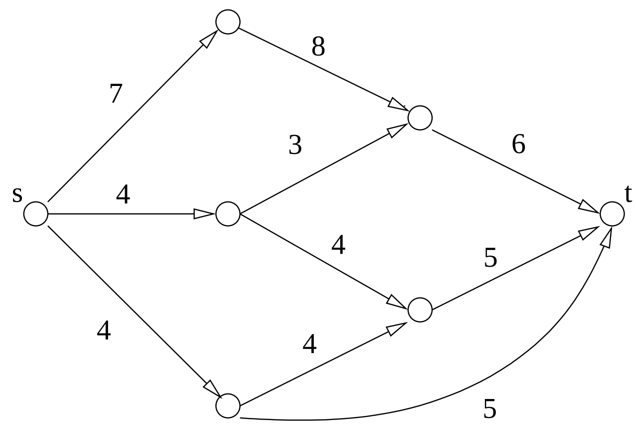




You can write your flow on the figure below.



(b) Give a minimum cut separating s from t. What is its capacity?



2. Consider the complete graph K_4 on 4 vertices (and thus 6 edges). Consider the matching polytope P, i.e. the convex hull of incidence vectors of matchings in K_4.

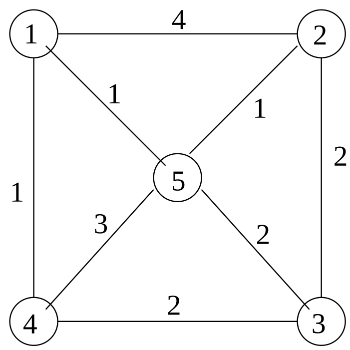
   (a) What is the dimension of P? Why?

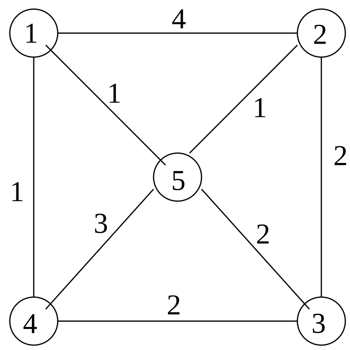
   (b) Define what a facet of a polyhedron is in general (it might be good to first say what a face is).

(c) Let x_{ij} denote the variable corresponding to the edge (i, j). Does the inequality $x_{12} + x_{13} + x_{23} \leq 1$ define a facet of the matching polytope P? Justify.

(d) Write down explicitly all linear inequalities defining P (for the graph K_4).

3. Consider the following undirected graph; edge capacities are indicated along every edge. Find the cut of minimum capacity. Show your work. You can use any algorithm that you would also use if the instance had 1000 vertices (so complete enumeration is out of the question).




1
2
3
4
5
4
1
1
2
1
3
2
2

4. (a) Define what a circuit C of a matroid $M = (E, \mathcal{I})$ is. Also state (without proof) the properties that the circuits of a matroid satisfy.

   (b) Based on what you wrote above, prove that if a set I is independent in a matroid ($I \in \mathcal{I}$) and $e \notin I$ then there exists $f \in I$ such that $I \setminus \{f\} \cup \{e\} \in \mathcal{I}$.

(c) Consider the following local search algorithm for the maximum weight spanning tree in a weighted undirected graph $G = (V, E)$. Start from any tree. At any point, define the neighborhood $N(T)$ of a tree T to be those trees that can be obtained from T by adding an edge in $V \setminus T$ and removing an edge of T (so as to maintain a spanning tree). Keep replacing the tree with a maximum weight tree in its neighborhood.

Is this an *exact* neighborhood, in the sense that whenever this local search algorithm terminates, we are guaranteed to have a maximum spanning tree? Explain. (There are several ways to approach this; one possibility is to think about the *exchange graph*.)

5. Summarize below what the ellipsoid algorithm is (without giving precise formulas), the main step in its analysis, and why it is important for combinatorial optimization. Be concise; there is no need to get into details or to precisely define things.

6. Given an undirected graph $G = (V, E)$, we would like to find the minimum number of colors needed to color the edges such that no cycle is monochromatic (i.e. has all its edges colored with the same color).

   (a) How many colors does one need for the complete graph on 8 vertices?

(b) *Hard.* For a general graph, show how one can reduce the problem of deciding whether k colors are enough (to properly color the graph as stated) to a (maximum cardinality) matroid intersection problem.

(c) *Bonus.* Give a minmax formula for the minimum number of colors one needs in general.