Massachusetts Institute of Technology Michel X. Goemans

18.409: Topics in TCS: Embeddings of Finite Metric Spaces

September 25, 2006 Due October 11th, 2006.

Exercises 1.

This is due by October 11th. You should first try to solve the problems on your own, and then you are welcome to discuss them with others or to read papers that may help.

1. Consider any tree metric (i.e.the shortest path metric of a tree) on n points. Show that it isometrically embeds into $l_\infty^{O(\log n)}$.

2. (a) Show that an n-point metric in l_1^d isometrically embeds into $l_\infty^{2^d}$ (thus, in this embedding the dimension is independent of the number of points in the metric space).

   (b) Deduce from this that the diameter of this set of points can be found in $O(d2^d n)$. (This is of course interesting only if the dimension d is small; for example, for constant dimension, this gives a linear-time algorithm.)

3. Consider the diamond graphs $\{D_m\}_{m=1}^{\infty}$ (see the scribe notes of Lecture 3 for an exact definition).

   (a) Show that the corresponding metric can be embedded into l_1 with constant distortion (distortion 2 is achievable, but any constant is fine).

   (b) Show a lower bound $c > 1$ on the distortion needed to embed the diamond graph into l_1. (I do not know what is the best c that can be proved :-)

4. For a Frechet embedding μ and any $p \geq 1$, prove the following l_p analogue to the lemma we proved in lecture for l_2 embeddings:

   If for all $x, y \in X$,
   $$d(x,y) \leq \gamma E_\mu[|d(x,A) - d(y,A)|]$$
   then the mapping $f : X \to \mathbb{R}^{2^n}$ embeds (X, d) into l_p with distortion γ.

5. In this exercise, you will show that $\alpha(G) = \beta(G)$ when $k = 2$. Recall the setting. We have a multicommodity flow problem in an undirected graph $G = (V, E)$ with $k = 2$ commodities with demands D_1 and D_2 between (s_1, t_1) and (s_2, t_2), and capacities $c : E \to \mathbb{R}_+$. $\alpha(G)$ represents the largest fraction of the demands that can be simultaneously satisfied, i.e. one can find a flow of value $\alpha(G)D_1$ between s_1 and t_1 and a flow of value $\alpha(G)D_2$ between s_2 and t_2. $\beta(G)$ on the other hand is the sparsest cut, thus $\beta(G) = \min(C_1/D_1, C_2/D_2, C_{12}/(D_1 + D_2))$ where C_i (resp. C_{12}) is the smallest capacity of a cut separating s_i from t_i (resp. a cut separating both s_1 from t_1 and s_2 from t_2).

   As a hint, consider two separate (single-commodity) flow problems (for which we know that max flow = min cut). The first flow problem is defined on $(V \cup \{(s,t)\}, E \cup \{(s,s_1), (s,s_2), (t,t_1), (t,t_2)\})$ with the capacities of the new edges being $\beta(G)D_1$ for (s, s_1) and (t, t_1) and $\beta(G)D_2$ for $(s, s_2), (t, t_2)$. The second flow problem is defined on $(V \cup \{(s,t)\}, E \cup \{(s,s_1), (s,t_2), (t,t_1), (t,s_2)\})$ with appropriate capacities. Show how to combine the flows of these two problems to deduce that $\alpha(G) = \beta(G)$.