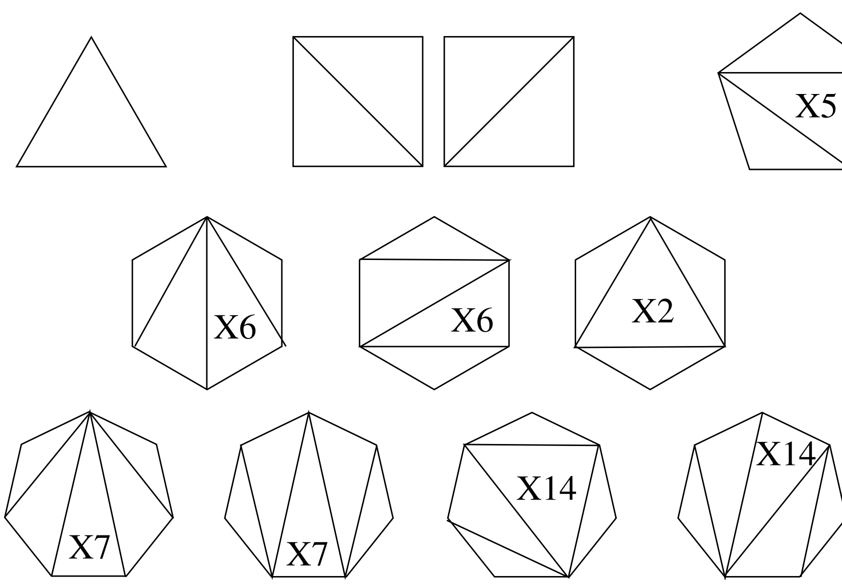
Spring 2015

18.310A Homework 2

Due Wed February 25th at 10AM in lecture

Instructions: Collaboration on homework is permitted, but you must write the solutions yourself; no copying is allowed. Please list the names of your collaborators; if you worked alone, state this. Also indicate any sources you consulted beyond the lecture notes.

1. Let C_k be the number of ways of putting chords in a $(k+2)$-gon in order to divide into triangles. Observe that $C_1 = 1$, $C_2 = 2$ (there are two ways of dividing a square into 2 triangles), and the figure below shows all the ways of placing chords in a j-gon for $j \leq 7$ and the number of times each should be counted because of symmetry. This shows that $C_3 = 5$, $C_4 = 14$ and $C_5 = 42$.



   Give a bijective proof that C_k is equal to the kth Catalan number. Carefully argue that your map is indeed a bijection. (You can use the fact that Catalan numbers count the number of Dyck paths, the number of plane trees or the number of binary trees.)

2. **Update on 2/20/2015: This question is not to be handed with problem set 2 on Feb 25th. Only questions 1, 3 and 4 need to be handed in.** Suppose $A_1, A_2, \cdots, A_k$ are subsets of cardinality n of a finite set X. We would like to color the elements of X red or blue in such that a way that in every A_i for $i = 1, \cdots, k$, there exists at least one red element and at least one blue element. Give a condition on k (as a function of n) such that this is always possible. (For maximum credit give the greatest function of n for which you can prove the result.)

3. (a) The classroom that we are in has six blackboard frames. In some of the lectures, the instructor enjoys showing his (lack of) drawing skills and draws a pigeon on one or several board frames. Show that over the course of a semester with 36 lectures, there exist two lectures and three board frames such that these three frames either all had no pigeons drawn on them in both lectures, or all had at least one pigeon drawn on each of them in both lectures.

   (b) Replace 36 now by k. Find the smallest k for which the above is still true. (In order to argue that your value of k is the smallest, you need to show that this is not true with $k - 1$.)

4. Informally, this question asks you to show that any real number (even irrational) can be approximated by a fraction with small denominator without incurring too much error. More formally, prove that for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N}$, there exist integers p and q with $1 \le q \le n$ satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{qn}.$$

   Hint: Prove that there exist p and q with $1 \le q \le n$ such that $|qx - p| < \frac{1}{n}$.