18.310 Exam 1 practice questions

This is a collection of problems from past quizzes and other sources. It does not necessarily (even certainly) reflect what will be on the exam on Monday. There will be *many fewer* questions.

1. Suppose one has two coins C_1 and C_2. Coin C_1 gives heads with probability $1/2$, while coin C_2 gives heads with probability $1/3$. We pick one of the coins uniformly at random and toss it twice. We get twice the same result. Compute the probability the it was coin C_1 being used.

2. You have a biased coin that comes up heads with probability $1/3$. Show that the probability of obtaining 80 heads or more from 90 throws is not more than 0.16.

3. In a permutation of n elements, a pair (j, i) is called an inversion if and only if $i < j$ and i comes after j. For example, the permutation 31542 in the case $n = 5$ has five inversions: (3,1), (3,2), (5,4), (5,2), and (4,2). What is the expected number of inversions in a uniform random permutation of the numbers $1, 2, \cdots, n$?

4. Prove that if C is any subset of $\{100, 101, \ldots, 199\}$ with $|C| = 51$, then C contains two consecutive integers.

5. You have an $n \times 3$ strip of unit squares. Let a_n be the number of ways you can tile it with 1×1 and 3×3 squares. Find the generating function $A(x) = \sum_{n \geq 0} a_n x^n$.

6. Consider the generating function
$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$
Suppose
$$f(x) = \frac{2 + 2x}{1 - 2x - x^2}.$$
Give an expression for a_n. Can you give a recursion for a_n and initial conditions that would give rise to this generating function?

7. A chess board is an eight-by-eight square grid, and a rook is a piece that can attack anything in the same row or column. Compute the number of ways of placing six rooks on a eight-by-eight chess board such that no two rooks attack each other.

8. Give a (short) expression for the number of Dyck walks of length $2n$ that starts with three steps up for all $n \geq 3$.

9. Let k be a fixed positive integer. For each nonnegative integer n, let b_n be the number of solutions to
$$x_1 + x_2 + \cdots + x_k = n,$$
where $x_i \in \{0, 3\}$ for all $1 \leq i \leq k$. Determine the (ordinary) generating function for $(b_n)_{n \in \mathbb{N}}$, and from this determine an explicit forumla for b_n.

10. Consider the following game that Harry and Tom play. They toss coins until they see three consecutive tosses that are either HHT or HTT, in that order. If they see HHT, Harry wins. If they see HTT, Tom wins. For example, if they toss TTHTHTT, Tom wins.

    (a) Conditioned on the first flip being H, what is the probability that Harry wins before they see two consecutive flips that are TH, in that order? Conditioned on the first flip being H, what is the probability that Tom wins before they see two consecutive flips that are TH, in that order? Note that if Harry wins first, Tom can't win, and vice versa.

    (b) Assume they play until one of them wins. What is the probability that Tom wins the game?

11. The projective plane of order 11 is a mathematical structure with 133 lines and 133 points. Any two lines intersect in one point, and any two points determine one line. There are exactly 12 points on each line, and each point lies on exactly 12 lines.

    Show that there is a coloring of the points of this projective plane with the colors red and blue such that every line has at least two points of each color.

12. Suppose you have a bag with the same number of red, green, orange and blue balls. Suppose you repeatedly draw a ball at random from the bag, observe its color, and place it back in the bag. Let Y be the number of red balls you get out of $n = 400$ drawings. It may be convenient to let X_i be 1 if the ith ball is red, and 0 otherwise (for $i \in \{1, \cdots, n\}$).

    (a) What are $\mathbb{E}[Y]$ and $\text{Var}(Y)$?

    (b) Derive an upper bound on $\mathbb{P}[Y \geq 200]$ using (i) Markov's inequality, (ii) Chebyshev's inequality, (iii) the Chernoff bound.