THE THEORY OF NONLINEAR SCHRÖDINGER EQUATIONS: PART I

J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO

Contents

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1. Introduction

The title of these lecture notes is certainly too ambitious. In fact here we will mainly consider semilinear Schrödinger initial value problems (IVP)

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $\lambda = \pm 1$, $p > 1$, $u : \mathbb{R} \times M \to \mathbb{C}$, and M is a manifold[1]. Even in this relatively special case we will not be able to mention all the findings and results concerning the initial value problem (1) and for this we apologize in advance.

Schrödinger equations are classified as *dispersive* partial differential equations and the justification for this name comes from the fact that if no boundary conditions are imposed their solutions tend to be waves which spread out spatially. But what does this mean mathematically? A simple and complete mathematical characterization of the word *dispersion* is given to us for example by R. Palais in [64]. Although his definition is given for one dimensional waves, the concept is expressed so clearly that it is probably a good idea to follow almost[2] literally his explanation: *"Let us [next] consider* linear *wave equations of the form*

$$u_t + P\left(\frac{\partial}{\partial x}\right) u = 0,$$

where P is polynomial. Recall that a solution $u(x,t)$, which Fourier transform is of the form $e^{i(kx-\omega t)}$, is called a plane-wave solution; k is called the wave number *(waves per unit of length) and ω the (angular)* frequency*. Rewriting this in the form $e^{ik(x-(\omega/k)t)}$, we recognize that this is a traveling wave of velocity $\frac{\omega}{k}$. If we substitute this $u(x,t)$ into our wave equation, we get a formula determining a unique frequency $\omega(k)$ associated to any wave number k, which we can write in the form*

$$\frac{\omega(k)}{k} = \frac{1}{ik}P(ik). \tag{2}$$

This is called the "dispersive relation" for this wave equation. Note that it expresses the velocity for the plane-wave solution with wave number k. For example, $P(\frac{\partial}{\partial x}) = c\frac{\partial}{\partial x}$ gives the linear advection equation $u_t + cu_x = 0$, which has the dispersion relation $\frac{\omega}{k} = c$, showing of course that all plane-wave solutions travel at the same velocity c, and we say that we have trivial dispersion in this case. On the other hand if we take $P\left(\frac{\partial}{\partial x}\right) = -\frac{i}{2}\left(\frac{\partial}{\partial x}\right)^2$, then our wave equation is $iu_t + \frac{1}{2}u_{xx} = 0$, which is the linear Schrödinger equation, and we have the non-trivial dispersion relation $\frac{\omega}{k} = \frac{k}{2}$. In this case, plane waves of large wave-number (and hence high frequency) are traveling much faster than low-frequency waves. The effect of this is to "broaden a wave packet". That is, suppose our initial condition is $u_0(x)$. We can use the Fourier transform[3] to write u_0 in the form

$$u_0(x) = \int \widehat{u_0}(k) e^{ikx}\, dk,$$

[1] In most cases M is the euclidian space $\mathbb{R}^n$ and only at the end we will mention some results and references when M is a different kind of manifold.

[2] R. Palais actually uses the Airy equation as an example, while we use the linear Schrödinger equation to be consistent with the topic of the lectures.

[3] In these lectures we will ignore the absolute constants that may appear in other definitions for the Fourier transform.

and then, by superposition, the solution to our wave equation will be

$$u(x,t) = \int \widehat{u_0}(k) e^{ik(x-(\omega(k)/k)t)}\, dk.$$

Suppose for example that our initial wave form is a highly peaked Gaussian. Then in the case of the linear advection equation all the Fourier modes travel together at the same speed and the Gaussian lump remains highly peaked over time. On the other hand, for the linearized Schrödinger equation the various Fourier modes all travel at different velocities, so after time they start cancelling each other by destructive interference, and the original sharp Gaussian quickly broadens".

As one can imagine dispersive equations are proposed as descriptions of certain phenomena that accour in nature. But it turned out that some of these equations appear also in more abstract mathematical areas like algebraic geometry [44], and certainly we are not in the position to discuss this beautiful part of mathematics here.

The questions that we will address here are more phenomenological. Assume that a profile of a wave is given at time $t = 0$, (initial data). Is it possible to prove that there exists a unique wave that "lives" for an interval of time $[0, T]$, that satisfies the equation, and that at time $t = 0$ has the assigned profile? What kind of properties does the wave have at later times? Does it "live" for all times or does it "blow up" in finite time?

Our intuition tells us that, if we start with *nice* and *small* initial data, then all the questions above should be easier to answer. This is indeed often true. In general in this case one can prove that the wave exists for all times, it is unique and its "size", measured taking into account the order of smoothness, can be controlled in a reasonable way. But what happens when we are not in this advantageous setting? These lecture notes are devoted to the understanding of how much of the above is still true when we consider *large* data and *long* interval of times. To be able to give a rigorous setting for the study of the initial value problem in (1) and to avoid any confusion in the future we need a strong mathematical definition for *well-posedness*. We consider the general initial value problem of type

$$\begin{cases} \partial_t u + P_m(\partial_{x_1}, \dots, \partial_{x_n})u + N(u, \partial_x^\alpha u) = 0, \\ u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n(\text{ or } x \in \mathbb{T}^n),\ t \in \mathbb{R}, \end{cases} \tag{3}$$

where $m \in \mathbb{N}$, $P_m(\partial_{x_1}, \dots, \partial_{x_n})$ is a differential operator with constant coefficients of order m and $N(u, \partial_x^\alpha u)$ is the nonlinear part of the equation, that is a nonlinear function that depends on u and derivatives of u up to order $m-1$. The function $u_0(x)$ is the initial condition or initial profile, and most of the time is called initial data. Above we pointed out the fact that finding a solution for an IVP strongly depends on the regularity one asks for the solution itself. So we first have to decide how we "measure" the regularity of a function. The most common way of doing so is to decide where the weak derivatives of the function "live". It is indeed time to recall the definition of Sobolev spaces[4]

Definition 1.1. We say that a function $f \in H^k(\mathbb{R}^n), k \in \mathbb{N}$ if f and all its partial derivatives up to order k are in L^2. We recall that $H^k(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{H^k} = \sum_{|\alpha|=0}^{k} \|\partial_x^\alpha f\|_{L^2},$$

where $\alpha(\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is its length.

We also recall here the definition of the Fourier transform.

[4]In more sophisticated instances one replaces Sobolev spaces with different ones, like L^p spaces, Hölder spaces, and so on.

Definition 1.2. Assume $f \in L^2(\mathbb{R}^n)$, then the Fourier transform of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(x)\, dx$$

where $\langle\cdot\rangle$ is the inner product in $\mathbb{R}^n$. We also have an inverse Fourier formula

$$f(x) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \hat{f}(x)\, dx.$$

If the function is defined on the torus $\mathbb{T}^n$ then the Fourier transform is defined as

$$\hat{f}(k) = \int_{\mathbb{T}^n} e^{i\langle x,k\rangle} f(x)\, dx$$

and the inverse Fourier formula is

$$f(x) = \sum_{k\in\mathbb{Z}} e^{-i\langle x,k\rangle} \hat{f}(k).$$

Remark 1.3. Because $\widehat{\partial_x^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$, it is easy to see that $f \in H^k(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2k}\, d\xi < \infty,$$

and moreover

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2k}\, d\xi\right)^{1/2} \sim \|f\|_{H^k}.$$

Then we can generalize our notion of Sobolev space and define $H^s(\mathbb{R}^n), s \in \mathbb{R}$ as the set of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s}\, d\xi < \infty.$$

Also $H^s(\mathbb{R}^n)$ is a Banach space with norm

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s}\, d\xi\right)^{1/2} \sim \|f\|_{H^s}.$$

Sometimes it is useful to use the *homogeneous* Sobolev space $\dot{H}^s(\mathbb{R}^n)$. This is the space of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s}\, d\xi < \infty.$$

Clearly all these observations can be made for Sobolev spaces in $\mathbb{T}^n$, except that in this case $\dot{H}^s(\mathbb{T}^n)$ and $H^s(\mathbb{T}^n)$ coincides.

We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^n)$ norm. We often need mixed norm spaces, so for example, we say that $f \in L^p_x L^q_t$ if $\|(\|f(x,t)\|_{L^q_t})\|_{L^p_x} < \infty$. Finally, for a fixed interval of time $[0,T]$ and a Banach space of functions Z, we denote with $C([0,T],Z)$ the space of continuous maps from $[0,T]$ to Z.

We are now ready to give a first definition of well-posedness. We will give a more refined one later in Subsection 3.10.

Definition 1.4. We say that the IVP (3) is locally well-posed (l.w.p) in H^s if, given $u_0 \in H^s$, there exist T, a Banach space of functions $X_T \subset C([-T,T];H^s)$ and a unique $u \in X_T$ which solves (3). Moreover we ask that there is continuity with respect to the initial data in the appropriate topology. We say that (3) is globally well-posed (g.w.p) in H^s if the definition above is satisfied in any interval of time $[-T,T]$.

Remark 1.5. The intervals of time are symmetric about the origin because the problems that we study here are all time reversible (i.e. if $u(x,t)$ is a solution, then so is $-u(x,-t)$).

We end this introduction with some notations. Throughout the notes we use C to denote various constants. If C depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C_{\|u_0\|_2}$ will depend on $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is an absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a+\varepsilon$ and $a-\varepsilon$, for some $0 < \varepsilon \ll 1$.

2. Lecture # 1: The Linear Schrödinger Equation in $\mathbb{R}^n$: Dispersive and Strichartz Estimates

In this lecture we introduce some of the most important estimates relative to the linear Schrödinger IVP

$$\begin{cases} iv_t + \frac{1}{2}\Delta v = 0, \\ v(x,0) = u_0(x). \end{cases} \tag{4}$$

It is important to understand as much as possible the solution v of (4) that we will denote with $v(x,t) = S(t)u_0(x)$, since by the Duhamel principle one can write the solution of the associated forced or nonlinear problem

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = F(u), \\ u(x,0) = u_0(x). \end{cases} \tag{5}$$

as

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')F(u(t'))\,dt'. \tag{6}$$

Problem 2.1. *Prove the Duhamel Principle* (6)*.*

The solution of the linear problem (4) is easily computable by taking Fourier transform. In fact by fixing the frequency ξ problem (4) transforms into the ODE

$$\begin{cases} i\hat{v}_t(t,\xi) - \frac{1}{2}|\xi|^2\hat{v}(t,\xi) = 0, \\ \hat{v}(\xi,0) = \hat{u}_0(\xi) \end{cases} \tag{7}$$

and we can write its solution as

$$\hat{v}(t,\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi).$$

In general the solution $v(t,x)$ above is denoted by $S(t)u_0$, where $S(t)$ is called the Schrödinger group. If we define, in the distributional sense,

$$K_t(x) = \frac{1}{(\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}}$$

then we have

$$S(t)u_0(x) = e^{it\Delta}u_0(x) = u_0 \star K_t(x) = \frac{1}{(\pi i t)^{n/2}}\int e^{i\frac{|x-y|^2}{2t}}u_0(y)\,dy \tag{8}$$

Problem 2.2. *Prove, in the sense of distributions, that the inverse Fourier transform of* $e^{-i\frac{1}{2}|\xi|^2 t}$ *is* $K_t(x) = \frac{1}{(\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}}$.

As mentioned already

$$\widehat{S(t)u_0}(\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi), \tag{9}$$

and this last one can be interpreted as saying that the solution $S(t)u_0$ above is the adjoint of the Fourier transform restricted on the paraboloid $P = \{(\xi, |\xi|^2) \;\text{ for } \xi \in \mathbb{R}^n\}$. This remark, strictly linked to (8) and (9), can be used to prove a variety of very deep estimates for $S(t)u_0$, see for example [69]. For example from (8) we immediatelly have the so called *Dispersive Estimate*

$$\|S(t)u_0\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\|u_0\|_{L^1}. \tag{10}$$

From (9) instead we have the conservation of the homogeneous Sobolev norms[5]

$$\|S(t)u_0\|_{\dot{H}^s} = \|u_0\|_{\dot{H}^s}, \tag{11}$$

for all $s \in \mathbb{R}$. Interpolating (10) with (11) when $s = 0$ and using a so called TT^* argument one can prove the famous Strichartz estimates (see [19], [47], and [71] for some concise proofs, and [19] for a complete list of authors who contributed to the final version of the theorem below):

Theorem 2.3 (Strichartz Estimates for the Schrödinger operator)**.** *Fix $n \geq 1$. We call a pair (q, r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q, r, n) \neq (2, \infty, 2)$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the homogeneous Strichartz estimate*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{12}$$

and the inhomogeneous Strichartz estimate

$$\left\| \int_0^t S(t-t')F(t')\,dt' \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{13}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

To finish this lecture we would like to present a *rifined* bilinear Strichartz estimate due originally to Bourgain in [9] (see also [12]).

Theorem 2.4. *Let $n \geq 2$. For any spacetime slab $I_* \times \mathbb{R}^n$, any $t_0 \in I_*$, and for any $\delta > 0$, we have*

$$\begin{aligned} \|uv\|_{L^2_t L^2_x(I_*\times\mathbb{R}^n)} \leq C(\delta)&(\|u(t_0)\|_{\dot{H}^{-1/2+\delta}} + \|(i\partial_t + \Delta)u\|_{L^1_t \dot{H}^{-1/2+\delta}_x}) \\ &\times (\|v(t_0)\|_{\dot{H}^{\frac{n-1}{2}-\delta}} + \|(i\partial_t + \Delta)v\|_{L^1_t \dot{H}^{\frac{n-1}{2}-\delta}_x}). \end{aligned} \tag{14}$$

This estimate is very useful when u is high frequency and v is low frequency, as it moves plenty of derivatives onto the low frequency term. This estimate shows in particular that there is little interaction between high and low frequencies. One can also check easily that when $n = 2$ one recovers the $L^4_t L^4_x$ Strichartz estimate contained in Theorem 2.3 above.

Proof. We fix δ, and allow our implicit constants to depend on δ. We begin by addressing the homogeneous case, with $u(t) := e^{it\Delta}\zeta$ and $v(t) := e^{it\Delta}\psi$ and consider the more general problem of proving

$$\|uv\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot{H}^{\alpha_1}} \|\psi\|_{\dot{H}^{\alpha_2}}. \tag{15}$$

Scaling invariance for this estimate[6] demands that $\alpha_1 + \alpha_2 = \frac{n}{2} - 1$. Our first goal is to prove this for $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$. The estimate (15) may be recast using duality and renormalization as

$$\int g(\xi_1+\xi_2, |\xi_1|^2+|\xi_2|^2)|\xi_1|^{-\alpha_1}\widehat{\zeta}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi}(\xi_2)d\xi_1 d\xi_2 \lesssim \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|\zeta\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{16}$$

Since $\alpha_2 \geq \alpha_1$, we may restrict our attention to the interactions with $|\xi_1| \geq |\xi_2|$. Indeed, in the remaining case we can multiply by $(\frac{|\xi_2|}{|\xi_1|})^{\alpha_2-\alpha_1} \geq 1$ to return to the case under consideration. In fact, we may further restrict our attention to the case where $|\xi_1| > 4|\xi_2|$ since, in the other case, we can move the frequencies between the two factors and reduce to the case where $\alpha_1 = \alpha_2$, which

[5]We will see later that the L^2 norm is conserved also for the nonlinear problem (1).

[6]Here we use the fact that if v is solution to the linear Schrödinger equation, then $v_\lambda(x,t) = v(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is also solution.

can be treated by $L^4_{t,x}$ Strichartz estimates[7] when $n \geq 2$. Next, we decompose $|\xi_1|$ dyadically and $|\xi_2|$ in dyadic multiples of the size of $|\xi_1|$ by rewriting the quantity to be controlled as (N, Λ dyadic):

$$\sum_N \sum_\Lambda \int \int g_N(\xi_1 + \xi_2, |\xi_1|^2 + |\xi_2|^2) |\xi_1|^{-\alpha_1} \widehat{\zeta_N}(\xi_1) |\xi_2|^{-\alpha_2} \widehat{\psi_{\Lambda N}}(\xi_2) d\xi_1 d\xi_2.$$

Note that subscripts on g, ζ, ψ have been inserted to evoke the localizations to $|\xi_1 + \xi_2| \sim N, |\xi_1| \sim N, |\xi_2| \sim \Lambda N$, respectively. Note that in the situation we are considering here, namely $|\xi_1| \geq 4|\xi_2|$, we have that $|\xi_1 + \xi_2| \sim |\xi_1|$ and this explains why g may be so localized.

By renaming components, we may assume that $|\xi_1^1| \sim |\xi_1|$ and $|\xi_2^1| \sim |\xi_2|$. Write $\xi_2 = (\xi_2^1, \underline{\xi_2})$. We now change variables by writing $u = \xi_1 + \xi_2,\ v = |\xi_1|^2 + |\xi_2|^2$ and $dudv = J d\xi_2^1 d\xi_1$. A calculation then shows that $J = |2(\xi_1^1 \pm \xi_2^1)| \sim |\xi_1|$. Therefore, upon changing variables in the inner two integrals, we encounter

$$\sum_N N^{-\alpha_1} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} g_N(u,v) H_{N,\Lambda}(u, v, \underline{\xi_2}) du dv d\underline{\xi_2}$$

where

$$H_{N,\Lambda}(u, v, \underline{\xi_2}) = \frac{\widehat{\zeta_N}(\xi_1) \widehat{\psi_{\Lambda N}}(\xi_2)}{J}.$$

We apply Cauchy-Schwarz on the u, v integration and change back to the original variables to obtain

$$\sum_N N^{-\alpha_1} \|g_N\|_{L^2} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2} \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|\widehat{\zeta_N}(\xi_1)|^2 |\widehat{\psi_{\Lambda N}}(\xi_2)|^2}{J} d\xi_1 d\xi_2^1 \right]^{\frac{1}{2}} d\underline{\xi_2}.$$

We recall that $J \sim N$ and use Cauchy-Schwarz in the $\underline{\xi_2}$ integration, keeping in mind the localization $|\xi_2| \sim \Lambda N$, to get

$$\sum_N N^{-\alpha_1 - \frac{1}{2}} \|g_N\|_{L^2} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2 + \frac{n-1}{2}} \|\widehat{\zeta_N}\|_{L^2} \|\widehat{\psi_{\Lambda N}}\|_{L^2}.$$

Choose $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$ with $\delta > 0$ to obtain

$$\sum_N \|g_N\|_{L^2} \|\widehat{\zeta_N}\|_{L^2} \sum_{\Lambda \leq 1} \Lambda^\delta \|\widehat{\psi_{\Lambda N}}\|_{L^2}$$

which may be summed up, after using the Schwarz inequality, and the Plancherel theorem will give the claimed homogeneous estimate.

We turn our attention to the inhomogeneous estimate (14). For simplicity we set $F := (i\partial_t + \Delta)u$ and $G := (i\partial_t + \Delta)v$. Then we use Duhamel's formula (6) to write

$$u = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-t')\Delta} F(t')\, dt', \quad v = e^{i(t-t_0)\Delta} v(t_0) - i \int_{t_0}^t e^{i(t-t')\Delta} G(t').$$

[7] In one dimension $n = 1$, Lemma 2.4 fails when u, v have comparable frequencies, but continues to hold when u, v have separated frequencies; see [24] for further discussion.

We obtain[8]

$$\begin{aligned}\|uv\|_{L^2} &\lesssim \left\| e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t_0)\Delta}v(t_0)\right\|_{L^2} \\ &+ \left\| e^{i(t-t_0)\Delta}u(t_0)\int_{t_0}^{t} e^{i(t-t')\Delta}G(t')\,dt'\right\|_{L^2} + \left\| e^{i(t-t_0)\Delta}v(t_0)\int_{t_0}^{t} e^{i(t-t')\Delta}F(t')dt'\right\|_{L^2} \\ &+ \left\| \int_{t_0}^{t} e^{i(t-t')\Delta}F(t')dt' \int_{t_0}^{t} e^{i(t-t'')\Delta}G(x,t'')\,dt''\right\|_{L^2} \\ &:= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

The first term was treated in the first part of the proof. The second and the third are similar so we consider only I_2. Using the Minkowski inequality we have

$$I_2 \lesssim \int_{\mathbb{R}} \|e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t')\Delta}G(t')\|_{L^2}\,dt',$$

and in this case the theorem follows from the homogeneous estimate proved above. Finally, again by Minkowski's inequality we have

$$I_4 \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \|e^{i(t-t')\Delta}F(t')e^{i(t-t'')\Delta}G(t'')\|_{L^2_x}dt'dt'',$$

and the proof follows by inserting in the integrand the homogeneous estimate above. $\square$

Remark 2.5. In the situation where the initial data are dyadically localized in frequency space, the estimate (15) is valid [9] at the endpoint $\alpha_1 = -\frac{1}{2}, \alpha_2 = \frac{n-1}{2}$. Bourgain's argument also establishes the result with $\alpha_1 = -\frac{1}{2} + \delta, \alpha_2 = \frac{n-1}{2} + \delta$, which is not scale invariant. However, the full estimate fails at the endpoint.

Problem 2.6. *Conider the following two questions:*

(1) *Prove that the full estimate at the endpoint is false by calculating the left and right sides of (16) in the situation where* $\widehat{\zeta_1} = \chi_{R_1}$ *with* $R_1 = \{\xi : \xi_1 = Ne^1 + O(N^{\frac{1}{2}})\}$ *(where* e^1 *denotes the first coordinate unit vector),* $\widehat{\psi_2}(\xi_2) = |\xi_2|^{-\frac{n-1}{2}}\chi_{R_2}$ *where* $R_2 = \{\xi_2 : 1 \ll |\xi_2| \ll N^{\frac{1}{2}}, \xi_2 \cdot e^1 = O(1)\}$ *and* $g(u,v) = \chi_{R_0}(u,v)$ *with* $R_0 = \{(u,v) : u = Ne^1 + O(N^{\frac{1}{2}}), v = |u|^2 + O(N)\}$.

(2) *Use the same counterexample to show that the estimate*

$$\|u\overline{v}\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot H^{\alpha}_1}\|\psi\|_{\dot H^{\alpha}_2},$$

where $u(t) = e^{it\Delta}\zeta,\ v(t) = e^{it\Delta}\psi$*, also fails at the endpoint.*

[8] Alternatively, one can absorb the homogeneous components $e^{i(t-t_0)\Delta}u(t_0)$, $e^{i(t-t_0)\Delta}v(t_0)$ into the inhomogeneous term by adding an artificial forcing term of $\delta(t-t_0)u(t_0)$ and $\delta(t-t_0)v(t_0)$ to F and G respectively, where δ is the Dirac delta.

3. Lecture # 2: The Nonlinear Schrödinger Equation (NLS) in $\mathbb{R}^n$: Conservation Laws, Classical Morawetz and Virial Identity, Invariances for the Equation

In this section we consider the (NLS) IVP (1) and we formally talk about the solution $u(x,t)$ as an object that exists, is smooth etc. Of course to be able to use whatever we say here later we will need to work on making this formal assumption true!

Given an equation it is always a good idea to read as much as possible out of it. So one should always ask what are the rigid constraints that an equation imposes on its solutions a-priori. Here we will look at conservation laws (in this case integrals involving the solution that are independent of time), some inequalities (or monotonicity formulas) that a solution has to satisfy, symmetries and invariances that a solution to (1) can be subject to. All three of these elements are somehow related (see for example Noether's theorem [71]) and here we will not even attempt to discuss ALL the possible connections. It is true though that in describing these important features of the equation one often has to recall some basic principles/quantities coming from physics like conservation of mass, energy and momentum, the notion of density, interaction of particles, resonance etc.

3.1. **Conservation laws.** A simple way to interpret physically the function $u(x,t)$ solving a Schrödinger equation is to think about $|u(x,t)|^2$ as the particle density at place x and at time t. Then it shouldn't come as a surprise that the density, momentum and energy are conserved in time. More precisely if we introduce the *pseudo-stress-energy tensor* $T_{\alpha,\beta}$ for $\alpha,\beta = 0,1,...,n$ then

$$T_{00} = |u|^2 \quad \text{(mass density)} \tag{17}$$

$$T_{0j} = T_{j0} = Im(\bar{u}\partial_{x_j}u) \quad \text{(momentum density)} \tag{18}$$

$$T_{jk} = Re(\partial_{x_j}u\overline{\partial_{x_k}u}) - \frac{1}{4}\delta_{j,k}\Delta(|u|^2) + \lambda\frac{p-1}{p+1}\delta_{jk}|u|^{p+1} \quad \text{(stress tensor)} \tag{19}$$

then by using the equation one can show that

$$\partial_t T_{00} + \partial_{x_j}T_{0j} = 0 \quad \text{and} \quad \partial_t T_{j0} + \partial_{x_k}T_{jk} = 0 \tag{20}$$

for all $j,k = 1,...,n$.

Problem 3.2. *Prove* (20) *using the equation.*

The conservation laws summarized in (20) are said to be *local* in the sense that they hold pointwise in the physical space. Clearly by integrating in space and assuming that u vanishes at infinity one also has the conserved integrals

$$m(t) = \int T_{00}(x,t)\,dx = \int |u|^2(x,t)\,dx \quad \text{(mass)} \tag{21}$$

$$p_j(t) = -\int T_{0j}(x,t) = -\int Im(\bar{u}\partial_{x_j}u)\,dx \quad \text{(momentum).} \tag{22}$$

We observe here that the stress tensor in (19) is not conserved, but it plays an important role in some "sophisticated" monotonicity formulas involving the solution u. To obtain the conservation of energy $E(t)$ we need to remember that the total energy of a system at time t is

$$E(t) = K(t) + P(t)$$

the sum of kinetic and potential energy. In our case

$$K(t) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx \quad \text{and} \quad P(t) = \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx$$

and hence

$$E(t) = \frac{1}{2}\int |\nabla u|^2(x,t)\, dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\, dx = E(0). \tag{23}$$

We immediately observe that now the sign of λ plays a very important role since by picking $\lambda = -1$ one can produce a negative energy. We will discus this later in greater details.

Problem 3.3. *Prove the conservation of energy* (23) *by using the equation.*

As we will see, to have an a-priori control in time of an energy like in (23) when $\lambda = 1$ is an essential tool in order to prove that a solution exists for all times. But it is also true that often this is not sufficient. This is indeed the case when the problem is *critical* [9]. We need then other a-priori controls on norms for the solution u. This is the content of the next subsection.

3.4. **Viriel and Classical Morawetz Identities.** The Viriel identity was first introduced by Glassey [38] to show blow up for certain focusing ($\lambda = -1$) NLS problems. The classical[10] Morawetz identity was introduced instead by Morawetz in the context of the wave equations [62]. In the NLS case it was introduced by Lin and Strauss [59]. Morawetz type identities are particularly useful in the defocusing setting ($\lambda = 1$).

In general these identities are used in order to show that a positive quantity (often a norm) involving the solution u has a monotonic behavior in time. Monotonic quantities are used systematically in the context of elliptic equations and although both the Viriel and Morawetz estimates go back to the 70's only recently they have been used, together with their variations, in a surprisingly powerful way in the context of dispersive equations.

Suppose that a function $a(x)$ is measuring a particular quantity for our system[11] and we want to look at its overage value and in particular at its change in time. To do so we integrate $a(x)$ against the mass density tensor in (17) and we compute using (20) and integration by parts

$$\partial_t \int a(x)|u|^2(x,t)\, dx = \int \partial_{x_j} a(x) Im(\bar{u}\partial_{x_j} u)(t,x)\, dx. \tag{24}$$

At this stage there is no obvious sign for the right hand side of the equality. The integrals appearing above have special names. In fact we can introduce the following definition:

Definition 3.5. Given the IVP (1), we define the associated Virial potential

$$V_a(t) = \int a(x)|u(t,x)|^2 dx \tag{25}$$

and the associated Morawetz action

$$M_a(t) = \int \partial_{x_j} a(x) \mathrm{Im}(\overline{u}\partial_{x_j} u) dx. \tag{26}$$

By taking the second derivative in time and by using again (20), we obtain

$$\begin{aligned}\partial_t^2 V_a(t) &= \partial_t^2 \int a(x)|u|^2(x,t)\, dx = \partial_t M_a(t) = \int (\partial_{x_j}\partial_{x_k} a(x)) Re(\partial_{x_j} u \overline{\partial_{x_k} u}\, dx \\ &+ \frac{\lambda(p-1)}{p+1}\int |u(t,x)|^{p+1}\Delta a(x)\, dx - \frac{1}{4}|u|^2(x,t)\Delta^2 a(x)\, dx.\end{aligned}$$

Now let's make a particular choice for $a(x)$.

[9]The notion of criticality will be introduced below.

[10]Here we talk about *classical* Morawetz type identities in order to distinguish them from the Interaction Morawetz ones.

[11]For example $a(x)$ could represent the distance to a particular point, or the characteristic function of a particular domain.

- If $a(x) = |x|^2$, then $\Delta^2 a(x) = 0$ and $\Delta a(x) = 2n$ so

$$\partial_t^2 \int |x|^2|u|^2(x,t)\,dx = 4E + \frac{2\lambda}{p+1}[n(p-1)-4]\int |u|^{p+1}\,dx. \tag{27}$$

Remark 3.6. For example in the focusing case $\lambda = -1$, when $n = 3$ and $p > \frac{7}{3}$, if one starts with $E < 0$, then the function $f(t) = \int |x|^2|u|^2(x,t)\,dx$ is concave down and positive ($f'(t)$ is monotone decreasing). Hence there exists $T^* < \infty$ such that there the function cannot longer exists. This was in fact the original argument of Glassey to show the existence of blow up time for certain focusing NLS equations.

- If $a(x) = |x|$, then (24) becomes

$$\partial_t \int |x||u|^2(x,t)\,dx = \int Im(\bar{u}\frac{x}{|x|}\cdot\nabla u)(t,x)\,dx, \tag{28}$$

and from here

$$\begin{aligned} \partial_t M_{|x|} &= \partial_t \int Im(\bar{u}\frac{x}{|x|}\cdot\nabla u)(t,x)\,dx = \int \frac{|\nabla\!\!\!/ u(t,x)|^2}{|x|}\,dx \\ &+ \frac{2(n-1)(p-1)\lambda}{p+1}\int \frac{|u(t,x)|^{p+1}}{|x|}\,dx - \frac{1}{4}\int |u(x,t)|^2(\Delta^2|x|)\,dx, \end{aligned} \tag{29}$$

where $\nabla\!\!\!/ u := \nabla - \frac{x}{|x|}(\frac{x}{|x|}\cdot\nabla)$ denotes the angular gradient of u.

Problem 3.7. *Above we used $a(x) = |x|$ which is clearly non smooth at zero. Check that if we take $n \geq 3$ and we replace $|x|$ with $\sqrt{x^2+\epsilon^2}$ and let $\epsilon \to 0$, then the identity (29) is correct.*

One can then compute that for $n \geq 3$, $(\Delta^2|x|) \leq 0$ in the sense of distributions. As a consequence, in the defocusing case $\lambda = 1$, after integrating in time over an interval $[t_0, t_1]$ one has

$$\int_{t_0}^{t_1} \frac{|\nabla\!\!\!/ u(t,x)|^2}{|x|}\,dx,\ \int_{t_0}^{t_1}\int \frac{|u(t,x)|^{p+1}}{|x|}\,dx \lesssim \sup_{[t_0,t_1]}\left|\int Im(\bar{u}\frac{x}{|x|}\cdot\nabla u)(t,x)\,dx\right|. \tag{30}$$

One can easily estimate the right hand side as

$$\sup_{[t_0,t_1]}\left|\int Im(\bar{u}\frac{x}{|x|}\cdot\nabla u)(t,x)\,dx\right| \lesssim \|u_0\|_{L^2}E^{1/2}$$

by using both conservation of mass and energy. But if less regularity is preferable then one can use the Hardy inequality (see Lemma A.10 in [71]) as in Lemma 6.9 that will be introduced later in Lecture #5, to obtain

$$\int_{t_0}^{t_1} \frac{|\nabla\!\!\!/ u(t,x)|^2}{|x|}\,dx,\ \int_{t_0}^{t_1}\int \frac{|u(t,x)|^{p+1}}{|x|}\,dx \lesssim \sup_{[t_0,t_1]}\|u(t)\|^2_{H^{1/2}}, \tag{31}$$

where now the disadvantage is the fact that the $H^{1/2}$ norm of u is not uniformly bounded in time.

3.8. **Invariances and symmetries.** In this section we only list invariances and symmetries but we do not attempt to describe their usefulness and applications except for one of them that we will start using in today's lecture.

(1) **Scaling Symmetry:** If u solve the IVP (1) then

$$u_\mu(x,t) = \mu^{-\frac{2}{p-1}} u\left(\frac{t}{\mu^2}, \frac{x}{\mu},\right) \quad \text{and} \quad u_{\mu,0}(x) = \mu^{-\frac{2}{p-1}} u\left(\frac{x}{\mu},\right) \tag{32}$$

solves the IVP for any $\mu \in \mathbb{R}$.

(2) **Galilean Invariance:** If u is again a solution to (1) then

$$e^{ix\cdot v} e^{it|v|^2/2} u(t, x - vt) \quad \text{with initial data} \quad e^{ix\cdot v} u_0(x)$$

for every $v \in \mathbb{R}^n$ also solves the same IVP.

(3) **Obvious Symmetries:** Time and space translation invariance, spatial rotation, phase rotation symmetry $e^{i\theta}u$, time reversal.

(4) **Pseudo-conformal Symmetry:** In the case $p = 1 + \frac{4}{n}$, if u is solution for (1) then also

$$\frac{1}{|t|^{n/2}} u\left(\frac{1}{t}, \frac{x}{t}\right) e^{i|x|^2/2t} \tag{33}$$

for $t \neq 0$ is solution to the same equation.

We now concentrate on the **scaling symmetry** and we show how this can be used to understand for which nonlinerity (or for which $p > 1$) the problem of well-posedness is most difficult to address.

If we compute $\|u_{\mu,0}\|_{\dot{H}^s}$ we see that

$$\|u_{\mu,0}\|_{\dot{H}^s} \sim \mu^{-s+s_c} \|u_0\|_{\dot{H}^s}, \tag{34}$$

where

$$s_c = \frac{n}{2} - \frac{2}{p-1}.$$

From (34) it is clear that if we take $\mu \to +\infty$ then

(1) if $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the interval of time is made longer: our intuition says that this is the best possible setting for well-posedness,

(2) if $s = s_c$ (**critical case**) the norm is invariant while the interval of time is made longer. This looks like a problematic situation.

(3) if $s < s_c$ (**super-critical case**) the norms grow as the time interval gets longer. Scaling is obviously against us.

In order to have a better intuition for scaling that also relates the dispersive part of the solution Δu with the nonlinear part of it $|u|^{p-1}u$, we use an informal argument as in [71]. Let's consider a special type of initial wave u_0. We want u_0 such that its support in Fourier space is localized at a large frequency $N >> 1$, its support in space is inside a Ball of radius $1/N$ and its amplitude is A. Here we are making the assumption that scaling is the only symmetry that could interfere with a behavior that goes from linear to nonlinear, but in general this is not the only one. We have

$$\|u_0\|_{L^2} \sim AN^{-n/2}, \quad \|u_0\|_{\dot{H}^s} \sim AN^{s-n/2}.$$

If we want $\|u_0\|_{\dot{H}^s}$ small then we need to ask that $A \ll N^{n/2-s}$. Now under this restriction we want to compare the liner term Δu with the nonlinear part $|u|^{p-1}u$:

$$|\Delta u| \sim AN^2 \quad \text{while} \quad |u|^p \sim A^p.$$

From here if $AN^2 \gg A^p$ we believe that the linear behavior would win, alternatively the nonlinear one would. Putting everything together we have that

$$A^{p-1} \ll N^2 \quad \text{and} \quad A \ll N^{n/2-s} \Longrightarrow s > s_c \quad \text{(more linear)} \tag{35}$$

$$A^{p-1} \gg N^2 \quad \text{and} \quad A \gg N^{n/2-s} \Longrightarrow s < s_c \quad \text{(more nonlinear).} \tag{36}$$

As announced at the beginning the so called "scaling argument" presented here should only be used as a guide line since in delivering it we make a purely formal calculation. On the other hand in some cases ill-posedness results below critical exponent have been obtained (see for example [22, 23]).

Problem 3.9. *Prove the conservation of mass using Forier transform for the IVP* (1) *when* $n = 1$ *and* $p = 3$*.*

3.10. **Definition of well-posedness.** We conclude this lecture by giving the precise definition of local and global well-posedness for an initial value problem, which in this case we will specify to be of type (1).

Definition 3.11 (Well-posedness)**.** We say that the IVP (1) is *locally well-posed (l.w.p)* in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$ there exist a time T and a Banach space of functions $X \subset L^\infty([-T,T], H^s(\mathbb{R}^n))$ such that for each initial data $u_0 \in B$ there exists a unique solution $u \in X \cap C([-T,T], H^s(\mathbb{R}^n))$ for the integral equation

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')|u|^{p-1}u(t'))\,dt'. \tag{37}$$

Furthermore the map $u_0 \to u$ is continuous as a map from H^s into $C([-T,T], H^s(\mathbb{R}^n))$. If uniqueness is obtained in $C([-T,T], H^s(\mathbb{R}^n))$, then we say that local well -posedness is *unconditional.*

If this hold for all $T \in \mathbb{R}$ then we say that the IVP is *globally well-posed (g.w.p).*

Remark 3.12. Our notion of global well-posedness does not require that $\|u(t)\|_{H^s(\mathbb{R}^n)}$ remains uniformly bounded in time. In fact, unless $s = 0, 1$ and one can use the conservation of mass or energy, it is not a triviality to show such an uniform bound. This can be obtained as a consequence of scattering, when scattering is available. In general this is a question related to *weak turbulence theory.*

4. Lecture # 3: Local and global well-posedness for the $H^1(R^n)$ subcritical NLS

Our intuition suggests that if one assumes enough regularity then l.w.p. should be true basically for any $p > 1$. We do not prove this here but one can check this in [19, 71], or use the argument that we will present below and the fact that for $s > n/2$ the space H^s is an algebra to obtain this result directly. Here we consider instead the IVP (1) with a nonlinearity that is H^1 subcritical, that is $1 < p < 1 + \frac{4}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$. To prove l.w.p for $H^s(\mathbb{R}^n)$, the general strategy that we will follow is based on the contraction method. This method is based on these four steps:

(1) Definition of the operator

$$L(v) = \chi(t/T)S(t)u_0 + c\chi(t/T)\int_0^t S(t-t')|v|^{p-1}v(t'))\,dt'$$

where $\chi(r)$ denotes a smooth nonnegative bump even function, supported on $-2 \leq r \leq 2$ and satisfying $\chi(r) = 1$ for $-1 \leq r \leq 1$.

(2) Definition of a Banach space X such that $X \subset L^\infty([-T, T], H^s(\mathbb{R}^n))$.

(3) Proof of the fact that for any ball $B \subset H^s(\mathbb{R}^n)$, there exist T and a ball $B_X \subset X$ such that the operator L sends B_X into itself and it is a contraction there.

(4) Extension of the uniqueness result in B_X to a unique result in the whole space X.

We observe that the continuity with respect to the initial data will be a consequence of the fact that the solution is found through a contraction argument. In fact in this case we obtain way more than just continuity.

Problem 4.1. *Discuss the regularity of the map $u_0 \to u$ from H^s into $L^\infty([-T, T], H^s(\mathbb{R}^n))$ when l.w.p. is proved by contraction method.*

We state the main theorem (for a complete list of authors who contributed to the final version of this theorem see [19]):

Theorem 4.2. *Assume that $1 < p < 1 + \frac{4}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$. Then the IVP (1) is l.w.p in $H^s(\mathbb{R}^n)$ for all $s_c < s \leq 1$, where $s_c = \frac{n}{2} - \frac{2}{p-1}$. Moreover if the nonlinearity is algebraic, that is $n = 2, 3$ and $p = 3$, then there is persistence of regularity, that is if $u_0 \in H^m$, $m \geq 1$ then the solution $u(t) \in H^m(\mathbb{R}^n))$, for all t in its time of existence. If in (1) we assume that $\lambda = 1$ (defocusing) then the IVP is globally well-posed for $s = 1$.*

Here we prove a less general version of this theorem, namely that under the conditions given above on p there is g.w.p in H^1. We do not prove l.w.p. for $s_c < s \leq 1$ since we would need to introduce a product rule for fractional derivatives and it would become too technical.

Our starting point is the definition of a Banach space X based on the norms we introduced with the Strichartz estimates.

Definition 4.3. Assume $I = [-T, T]$ is fixed. The space $S^0(I \times \mathbb{R}^n)$ is the closure of the Schwartz functions under the norm

$$\|f\|_{S^0(I\times\mathbb{R}^n)} = \sup_{(q,r)\ n\text{-admissible}} \|f\|_{L^q_t L^r_x}.$$

We then define the space $S^1(I \times \mathbb{R}^n)$ where the closure is taken with respect to the norm

$$\|f\|_{S^1(I\times\mathbb{R}^n)} = \|f\|_{S^0(I\times\mathbb{R}^n)} + \|\nabla f\|_{S^0(I\times\mathbb{R}^n)}.$$

Proof. We consider the operator Lv and using (12) and (13) we obtain

$$\|Lv\|_{S^1(I\times\mathbb{R}^n)} \leq C_1\|u_0\|_{H^1} + C_2\||v|^{p-1}|\nabla v|\|_{L_t^{q'}L_x^{r'}}, \tag{38}$$

where (q,r) is a Strichartz admissible pair. The best couple to use in this context is the one that solves the system

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2} \quad \text{Strichartz Condition} \tag{39}$$

$$(p-1)\left(\frac{1}{r}-\frac{s}{n}\right)=\frac{1}{r'}-\frac{1}{r}, \tag{40}$$

and the meaning of the second equation will become clear below. The solutions to the system is

$$\frac{1}{r}=\frac{1}{(p+1)}+\frac{(p-1)}{(p+1)}\frac{s}{n} \quad \text{and} \quad \frac{1}{q}=\frac{(p-1)(n-2s)}{4(p+1)}.$$

From here it follows that[12]

$$\frac{1}{q'}>\frac{p}{q} \quad \Longrightarrow \quad s>s_c=\frac{n}{2}-\frac{2}{p-1}.$$

Then by Hölder inequality repeated

$$\||v|^{p-1}|\nabla v|\|_{L_t^{q'}L_x^{r'}} \leq T^\alpha\||v|^{p-1}|\nabla v|\|_{L_t^{q/p}L_x^{r'}} \leq \|\nabla v\|_{L_t^qL_x^r}\|v\|^{p-1}_{L_t^qL_x^{\tilde{r}}}$$

where $\frac{1}{\tilde{r}}=\frac{1}{r}-\frac{s}{n}$. By Sobolev embedding

$$\|v\|_{L_t^qL_x^{\tilde{r}}} \lesssim \|(1+\Delta^{\frac{s}{2}})v\|_{L_t^qL_x^r}, \tag{41}$$

and since we are assuming that we are in the H^1 subcritical regime $1<p<1+\frac{4}{n-2}$ it also follows that $s\leq 1$ and as a consequence

$$\||v|^{p-1}|\nabla v|\|_{L_t^{q'}L_x^{r'}} \leq T^\alpha\|v\|^p_{S^1}.$$

We can now conclude that

$$\|Lv\|_{S^1(I\times\mathbb{R}^n)} \leq C_1\|u_0\|_{H^1}+C_2T^\alpha\|v\|^p_{S^1}. \tag{42}$$

With similar arguments one also obtains

$$\|Lv-Lw\|_{S^1(I\times\mathbb{R}^n)} \leq C_2T^\alpha(\|v\|^{p-1}_{S^1}+\|w|^{p-1}_{S^1})\|v-w\|_{S^1}. \tag{43}$$

We are now ready to set up the contraction: pick $R=2C_1\|u_0\|_{H^1}$ and T such that

$$C_2T^\alpha R^{p-1}<\frac{1}{2} \quad \Longleftrightarrow \quad T\lesssim \|u_0\|_{H^1}^{\frac{1-p}{\alpha}}, \tag{44}$$

then clearly from (42), (43) and (44) it follows that $L: B_R\to B_R$, where B_R is the ball centered at zero and radius R in S^1, and L is a contraction. There is a unique fixed point $u\in B_R$ that is in fact a solution to our integral equation. The next two properties for u that we need to show are continuity with respect to time, that is $u\in C([-T,T],H^1)$ and uniqueness in the whole space S^1. The first is left to the reader since it is a simple consequence of the representation of u through the Duhamel formula (6). For the second we assume that there exists another solution $\tilde{u}\in S^1$ for the IVP (1). Using again the Duhamel formula for both u and $\tilde{u}$ and the estimates presented above for Lv we obtain that on an interval of time δ

$$\|u-\tilde{u}\|_{S^1_\delta} \leq C_2\delta^\alpha(\|\tilde{u}\|^{p-1}_{S^1_T}+\|u|^{p-1}_{S^1_T})\|u-\tilde{u}\|_{S^1_\delta}$$

[12] As mentioned above here we only address l.w.p. in H^1, but it is clear that if one uses fractional derivatives and (41) l.w.p in H^s, $s>s_c$ can also be obtained based on the fact that r and q are given in terms of s and $s>s_c$.

where here we use the lover index δ or T to stress that in the first case the space S^1 is relative to the interval $[-\delta, \delta]$ and in the second to $[-T, T]$. Since u and $\tilde{u}$ are fixed we can introduce

$$M = \max(\|\tilde{u}\|^{p-1}_{S^1_T} + \|u|^{p-1}_{S^1_T})$$

and if δ is small enough in terms of C_2, α and M we obtain

$$\|u - \tilde{u}\|_{S^1_\delta} \leq \frac{1}{2}\|u - \tilde{u}\|_{S^1_\delta}$$

which forces $u = \tilde{u}$ in $[-\delta, \delta]$. To cover the whole interval $[-T, T]$ then one iterates this argument $\frac{T}{\delta}$ times and the conclusion follows.

Before going to the proof of g.w.p we would like to consider the question of **propagation of regularity**. As mentioned above with this we mean the answer to the following question: assume that in (1), with the restrictions on p above, we start with $u_0 \in H^m,\ m \geq 1$. Is it true that the unique solution $u \in S^1$ also belongs to H^m at any later time $t \in [0, T]$? The answer to this depends on the regularity of the non-linear term, more precisely the regularity of the function $f(z) = |z|^{p-1}z$. This function is not C^∞ for all p, hence one cannot expect propagation of regularity for all p in the considered range. On the other hand if f is algebraic, namely when $p - 1 = 2k$ for some $k \in \mathbb{N}$, then propagation of regularity follows from the estimates we presented above. Briefly we can go back to (42) and if we repeat the same argument we obtain that for the solution u that we already found using only H^1 regularity we also have

$$\|D^m u\|_{S^0} \leq C_1\|u_0\|_{H^m} + C_2 T^\alpha \|u\|^{2k}_{S^1}\|D^m u\|_{S^0}$$

because when we apply the operator D^m the term with $D^m u$ appears linearly[13]. Since we already know that $C_2 T^\alpha \|u\|^{2k}_{S^1} \leq \frac{1}{2}$ we then obtain[14] that

$$\|D^m u\|_{S^0} \lesssim \|u_0\|_{H^m}.$$

We are now ready for the iteration of the local in time solution u to a uniformly global one[15]. The first step is to go back to (44) and notice that T depends on the H^1 norm of the initial data. From the previous lecture we learned that for a *smooth*[16] solution u to (1) the conservation of the energy and mass gives an a priori uniform bound

$$\|u(t)\|_{H^1} \leq C^*(\|u_0\|^1_H),$$

so if we take now $T^* \sim (C^*)^{\frac{1-p}{\alpha}}$ we can repeat the argument above with no changes. In particular when we get to time T^* we can apply the argument again with the new initial data $u(T^*)$ and the same T^* will work. In this way we can cover the whole time real lime and well-posedness becomes global. But in the argument we just outlined there is a *caviat* in the sense that if $u_0 \in H^1$ we do not have a *smooth* solution u. This obstacle can be overcome by introducing various smoothing tools. The precise argument can be found in [19]. □

[13] Here we are cheating a little since we are ignoring the mixed lower order derivatives. For this reason the constant C_2 is the same as the one in (42). If one does this calculation correctly then that constant C_2 will need to be replaced by a larger one, which will shrink the time T. To cover the whole interval $[-T, T]$ then one uses the iteration we introduced while proving uniqueness in S^1.

[14] Here we are cheating again in the sense that in principle we cannot even talk about $D^m u$ since we don't know yet that this expression makes sense. The rigorous procedure tells us to start with a smooth approximation of the initial data, the associated solution exists and is unique. Only at this point one can use the argument proposed here to get the uniform bound independent of the approximation.

[15] This argument only works when a uniform H^1 bound in time for the solution is available, for example in the defocusing case or when the L^2 norm of the initial data is small enough.

[16] Here with smooth we also mean zero at infinity.

Remark 4.4. We are not addressing in this first part of the course g.w.p. for the focusing NLS (1) even in the subcritical case. In order to address this issue we need to introduce stationary solutions (or solitons) and this will be done later. But assuming that the readers know about solitons and the Gagliardo-Niremberg inequality, then it is easy to see that l.w.p can be extended to g.w.p. as long as the mass of the initial data is strictly below the mass of the stationary solution. This condition in fact can be used with the Gagliardo-Niremberg inequality to show also in this case that the H^1 norm of the solutions remains uniformly bounded in time.

Remark 4.5. By carefully keeping track of the various exponents that have been introduced in order to get to (42) one can see that for the **critical** H^1 problem, that is $p = 1 + \frac{4}{n-2}$, the estimates are border line. In fact one gets

$$\|Lv\|_{S^1(I\times\mathbb{R}^n)} \leq C_1\|u_0\|_{H^1} + C_2\|v\|_{S^1}^p. \tag{45}$$

The main difference between this and (42) is that there is no time factor appearing in the right hand side. This of course makes the contraction more difficult to attain by shrinking the time. On the other hand if one starts with small data $\|u_0\|_{H^1} \leq \epsilon$ and calls now $R = 2C_1\epsilon$, then a sufficient condition on ϵ to have a contraction would be

$$C_2R^{p-1} = C_2(2C_1\epsilon)^{p-1} \leq \frac{1}{2}.$$

This would also guarantee a uniform global solution in H^1.

One could ask if at least l.w.p could be still achieved for large data. The following theorem gives a positive answer.

Theorem 4.6 (L.w.p. for H^1 critical NLS)**.** *Assume that $p = 1 + \frac{4}{n-2}$ and $u_0 \in H^1$. Assume also that there exists T such that*

$$\|S(t)u_0\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}} \leq \epsilon \tag{46}$$

for ϵ small enough. Then (1) *is H^1 well posed in $[-T,T]$.*

Proof. We first notice that the pair $(\frac{2(n+2)}{n-2}, \frac{2n(n+2)}{n^2+4})$ is Strichartz admissible. We define the new space $\tilde{S}^1$ using the following norm

$$\|f\|_{\tilde{S}^1} := T\|f\|_{S^1} + \|f\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}},$$

The idea is to use a contraction method in this space based on the smallness assumption (46). As we did in the proof of Theorem 4.2 we estimate Lv in the space $\tilde{S}^1$:

$$\begin{aligned}\|Lv\|_{\tilde{S}^1} &\lesssim T\|u_0\|_{H^1} + \|S(t)u_0\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}} \\ &+ \||v|^{\frac{4}{n-2}}|\nabla v|\|_{L^{\tilde{q}'}_{[-T,T]} L_x^{\tilde{r}'}}\end{aligned}$$

Now we pick the Strichartz pair $(\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})$ and we obtain by Hölder

$$\||v|^{\frac{4}{n-2}}|\nabla v|\|_{L^{\tilde{q}'}_{[-T,T]} L_x^{\tilde{r}'}} \lesssim \|v\|^{\frac{4}{n-2}}_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} L_x^{\frac{2(n+2)}{n-2}}} \|v\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}}.$$

By the Sobolev embedding theorem we then have

$$\|v\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} L_x^{\frac{2(n+2)}{n-2}}} \lesssim \|v\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}},$$

hence the final bound

$$\|Lv\|_{\tilde{S}^1} \lesssim T\|u_0\|_{H^1} + \|S(t)u_0\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}} + \|v\|^{1+\frac{4}{n-2}}_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}}. \tag{47}$$

Now if T is small enough, in particular $T \sim \epsilon\|u_0\|^{-1}_{H^1}$, using (46), we deduce from (47) that

$$\|Lv\|_{\tilde{S}^1} \leq 2C_0\epsilon + C_1\|v\|^{1+\frac{4}{n-2}}_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}}.$$

We then take a ball B of radius $R = 4C_0\epsilon$ and if ϵ is small enough then L sends B into itself and it is a contraction. The rest is now routine. This argument proved the theorem in the interval of time of length approximately $\epsilon\|u_0\|^{-1}_{H^1}$. In order to cover an arbitrary interval $[-T,T]$, then one has to use again the conservation of energy and mass that gives a uniform bound on $\|u\|_{H^1}$. $\square$

Remark 4.7. We have the following two facts:

(1) By the homogeneous Strichartz estimate (12) it follows that

$$\|S(t)u_0\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}} \lesssim \|u_0\|_{H^1}$$

hence we recover above the small data g.w.p we discussed in Remark 4.5.

(2) Given any data $u_0 \in H^1$, again by (12) we have

$$\|S(t)u_0\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} W_x^{1,\frac{2n(n+2)}{n^2+4}}} \leq C,$$

so we can use the time integral to claim that for T small enough (46) is satisfied. This gives l.w.p. but it is important to notice that in this case $T = T(u_0)$ depends also on the profile of the initial data, not only on its H^1 norm.

The next theorem gives a sort of criteria for the g.w.p. of the H^1 critical NLS. It says that if a certain Strichartz norm of the solution (actually any of them would do!) stays a-priori bounded, then g.w.p. follows.

Theorem 4.8 (G.w.p. for H^1 critical NLS with $L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}$ bound)**.** *Assume that $p = 1+\frac{4}{n-2}$ and $u_0 \in H^1$. Assume also the a priori estimate*

$$\|u\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} L_x^{\frac{2(n+2)}{n-2}}} \leq C \tag{48}$$

for any solution u to (1) *with $p = 1 + \frac{4}{n-2}$. Then this IVP is H^1 globally well posed.*

Proof. Fix ϵ to be determined later. Using (48) we can find finitely many intervals of time $I_1, ..., I_M$ such that

$$\|u\|_{L^{\frac{2(n+2)}{n-2}}_{I_j} L_x^{\frac{2(n+2)}{n-2}}} \leq \epsilon \tag{49}$$

for all $j = 1,..,M$. The goal here is to prove that as a consequence of (49) one actually has the stronger bound

$$\|u\|_{S^1_{I_j}} \leq C \tag{50}$$

for all $j = 1,..,M$ and putting all the intervals together

$$\|u\|_{S^1} \leq C \tag{51}$$

How do we use now this bound? We use it to continue the solution. More precisely: since $u_0 \in H^1$ we already proved that there exists T and a unique solution $u \in S^1_{[-T,T]}$. Let now T_{max} be the maximum time for well-posedness. Clearly if $T_{max} = +\infty$ there is nothing to prove. Suppose by contradiction that $T_{max} < +\infty$. We can use then (51) to claim in particular that

$$\|u(T_{max})\|_{H^1} \leq C.$$

Then we can continue our solution and obtain a contradiction.

It is now time to prove (50). Using estimates like the ones in the proof of Theorem 4.6 this time applied to the Duhamel representation of a solution u we have

$$\|u\|_{S^1_{I_j}} \leq C_1\|u_0\|_{H^1} + C_2\|u\|^{\frac{4}{n-2}}_{L^{\frac{2(n+2)}{n-2}}_{I_j} L_x^{\frac{2(n+2)}{n-2}}} \|u\|_{S^1_{I_j}} \leq C_1\|u_0\|_{H^1} + C_2\epsilon^{\frac{4}{n-2}}\|u\|_{S^1_{I_j}}$$

and if $C_2\epsilon^{\frac{4}{n-2}} < 1/2$ then (50) follows.

$\square$

We end this section by announcing that similar theorems, replacing H^1 with L^2 are available for the L^2 subcritical NLS, that is when $1 < p < 1 + \frac{4}{n}$. We do not list them here, but they can be found in [71].

5. Lecture # 4: Global well-posedness for the $H^1(R^n)$ subcritical NLS and the "I-method"

We learned during last lecture that for the H^1 subcritical NLS, i.e. $1 < p < 1+\frac{4}{n-2}$ and hence $s_c < 1$, l.w.p for (1), either focusing or defocusing, is available in $H^s(\mathbb{R}^n)$ for any $s, s_c \leq s \leq 1$. We also learned that if $s = 1$, in the defocusing case, uniform g.w.p is a consequence of the conservation of mass and energy. We then ask: if $0 \leq s_c < s < 1$ is the defocusing NLS problem globally well posed in H^s? This problem is particularly interesting when we consider the L^2 critical NLS, i.e. $s_c = 0$ and $p = 1 + \frac{4}{n}$. In this case the L^2 norm cannot be used to iterate the l.w.p. since the time interval of existence also depends on the profile of the initial data. It is clear then that this is a difficult question since we are in a regime when the conservation of the L^2 norm is too little of an information and the conservation of the Hamiltonian cannot be used since the data has not enough regularity. It was exactly to answer these kinds of questions that the "I-method" [24, 25, 26, 27, 48, 49] was invented. Unfortunately the method is quite technical to be applied in higher dimensions in its full strength. The results that we will report below are not optimal and in general they concern the L^2 critical case $p = 1 + \frac{4}{n}$ since that one is the most interesting, but similar results are available for the general H^1 subcritical case when $s_c < 1$ (see [20, 76]). We will list below the state of the art at this point for this problem for the L^2 critical case. We will give references but we will not prove these theorems in full generality. At the end of this lecture we will prove a weaker result than the one stated here when $n = 2$, see Theorem 5.2. We should also say here that if one assumes radial symmetry, then the L^2 critical NLS for $n \geq 2$ has been proved to be globally well-posed both in the defocusing and focusing case with the assumption that the mass of the initial data is strictly less than the mass of the stationary solution. These results are contained in a series of very recent and deep papers [57, 58, 73, 74], see also [56]. The point here is instead to address the question of global well-posedness without assuming radial symmetry and to present the "I-method".

Theorem 5.1 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n \geq 3$)**.** *The initial value problem* (1) *with $\lambda = 1$, $p = 1 + \frac{4}{n}$ is globally well-posed in $H^s(\mathbb{R}^n)$, for any $1 \geq s > \frac{\sqrt{7}-1}{3}$ when $n = 3$, and for any $1 \geq s > \frac{-(n-2)+\sqrt{(n-2)^2+8(n-2)}}{4}$ for $n \geq 4$.*

Here we have to assume that $s \leq 1$ since in general the non smoothness of the nonlinearity doesn't allow us to prove persistence of regularity. The proof of this theorem can be found in [34].

Theorem 5.2 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n = 2$)**.** *The initial value problem* (1) *with $\lambda = 1, n = 2$ and $p = 3$ is globally well-posed in $H^s(\mathbb{R}^2)$, for any $1 > s > \frac{2}{5}$. Moreover the solution satisfies*

$$\sup_{[0,T]} \|u(t)\|_{H^s} \leq C(1+T)^{\frac{3s(1-s)}{2(5s-2)}}, \tag{52}$$

where the constant C depends only on the index s and $\|u_0\|_{L^2}$.

Here the theorem is stated only for $s < 1$ since we already know that global well-posedness for $s \geq 1$ follows from conservation of mass and energy as explained in the previous lecture [17].

[17] It is an open problem to obtain a polynomial bound like in (52) for this problem when $s > 1$ and the data are not radial. In fact if if $p > 3$ a uniform bound follows from scattering. But scattering is still an open problem for general data for the L^2 critical NLS. We should also stress that these kinds of polynomial bounds for higher Sobolev norms are particularly interesting since they are related to the *weak turbulence theory*, a topic that we will not address here.

For the proof of Theorem 5.2 see [32]. The argument is based on a combination of the "I-metod" as in [25, 26, 28] and a refined two dimensional Morawetz interaction inequality. This combination first appeared in [37].

Finally we recall the result for the L^2 critical problem for $n = 1$:

Theorem 5.3 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n = 1$)**.** *The initial value problem* (1) *with $\lambda = 1, n = 1$ and $p = 5$ is globally well-posed in $H^s(\mathbb{R})$, for any $1 > s > \frac{1}{3}$. Moreover the solution satisfies*

$$\sup_{[0,T]} \|u(t)\|_{H^s} \leq C(1+T)^{\frac{s(1-s)}{2(3s-1)}}, \tag{53}$$

where the constant C depends only on the index s and $\|u_0\|_{L^2}$.

For the proof of this theorem see [36].

As promised we sketch now the proof of a weaker result than the one reported in Theorem 5.2, namely g.w.p. for $s > 4/7$. This proof is a summary of the work that appeared in [26]. Since below we will often refer to a particular IVP we write it here once for all

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = |u|^2 u, \\ u(x,0) = u_0(x). \end{cases} \tag{54}$$

To start the argument we need to introduce some notation and state some lemma.

We will use the weighted Sobolev norms,

$$||\psi||_{X_{s,b}} \equiv ||\langle\xi\rangle^s \langle\tau - |\xi|^2\rangle^b \tilde{\psi}(\xi,\tau)||_{L^2(\mathbb{R}^n\times\mathbb{R})}. \tag{55}$$

Here $\tilde{\psi}$ is the space-time Fourier transform of ψ. We will need local-in-time estimates in terms of truncated versions of the norms (55),

$$||f||_{X^\delta_{s,b}} \equiv \inf_{\psi = f \text{on}[0,\delta]} ||\psi||_{X^\delta_{s,b}}. \tag{56}$$

We will often use the notation $\frac{1}{2}+ \equiv \frac{1}{2} + \epsilon$ for some universal $0 < \epsilon \ll 1$. Similarly, we shall write $\frac{1}{2}- \equiv \frac{1}{2} - \epsilon$, and $\frac{1}{2} -- \equiv \frac{1}{2} - 2\epsilon$.

For a Schrödinger admissible pair (q,r) we have what we will call the $L^q_t L^r_x$ Strichartz estimate:

$$||\phi||_{L^q_t L^r_x(\mathbb{R}^{n+1})} \lesssim ||\phi||_{X_{0,\frac{1}{2}+}}, \tag{57}$$

which can be proved to be a consequence of (55).

Finally, we will need a refined version of these estimates due to Bourgain [9].

Lemma 5.4. *Let $\psi_1, \psi_2 \in X^\delta_{0,\frac{1}{2}+}$ be supported on spatial frequencies $|\xi| \sim N_1, N_2$, respectively. Then for $N_1 \leq N_2$, one has*

$$||\psi_1 \cdot \psi_2||_{L^2([0,\delta]\times\mathbb{R}^2)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} ||\psi_1||_{X^\delta_{0,\frac{1}{2}+}} ||\psi_2||_{X^\delta_{0,\frac{1}{2}+}}. \tag{58}$$

In addition, (58) *holds (with the same proof) if we replace the product $\psi_1 \cdot \psi_2$ on the left with either $\overline{\psi}_1 \cdot \psi_2$ or $\psi_1 \cdot \overline{\psi}_2$.*

This lemma is a consequence of the Theorem 2.4.

Problem 5.5. *Show how to deduce* (57) *and* (58)*.*

Hint: *Consider the space of frequencies both in time and space. Partition it into parabolic strips of approximate unit size. On each of these strips a function ψ can be viewed as a solution of the linear problem. Use the appropriate Strichartz or improved Strichartz on each of them and then sum with the appropriate weight.*

For rough initial data, with $s < 1$, the energy is infinite, and so the conservation law (23) is meaningless. Instead, here we use the fact that a smoothed version of the solution of the IVP (54) has a finite energy which is *almost* conserved in time. We express this 'smoothed version' as follows.

Given $s < 1$ and a parameter $N \gg 1$, define the multiplier operator

$$\widehat{I_N f}(\xi) \equiv m_N(\xi)\hat{f}(\xi), \tag{59}$$

where the multiplier $m_N(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

$$m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases} \tag{60}$$

For simplicity, we will eventually drop the N from the notation, writing I and m for (59) and (60). Note that for solution and initial data u, u_0 of (54), the quantities $||u||_{H^s(\mathbb{R}^n)}(t)$ and $E(I_N u)(t)$ (see (23)) can be compared,

$$E(I_N u)(t) \leq \left(N^{1-s}||u(\cdot,t)||_{\dot{H}^s(\mathbb{R}^2)}\right)^2 + ||u(t,\cdot)||^4_{L^4(\mathbb{R}^2)}, \tag{61}$$

$$||u(\cdot,t)||^2_{H^s(\mathbb{R}^2)} \lesssim E(I_N u)(t) + ||u_0||^2_{L^2(\mathbb{R}^2)}. \tag{62}$$

Indeed, the $\dot{H}^1(\mathbb{R}^2)$ component of the left hand side of (61) is bounded by the right side by using the definition of I_N and by considering separately those frequencies $|\xi| \leq N$ and $|\xi| \geq N$. The L^4 component of the energy in (61) is bounded by the right hand side of (61) by using (for example) the Hörmander-Mikhlin multiplier theorem. The bound (62) follows quickly from (60) and L^2 conservation (21) by considering separately the $\dot{H}^s(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ components of the left hand side of (62).

To prove our result, we may assume that $u_0 \in C_0^\infty(\mathbb{R}^2)$, and show that the resulting global-in-time solution grows at most polynomially in the H^s norm,

$$||u(\cdot,t)||_{H^s(\mathbb{R}^2)} \leq C_1 t^M + C_2, \tag{63}$$

where the constants C_1, C_2, M depend only on $||u_0||_{H^s(\mathbb{R}^2)}$ and not on higher regularity norms of the smooth data. The result then follows immediately from (63), the local-in-time theory discussed in the previous lecture, and a standard density argument.

By (62), it suffices to show

$$E(I_N u)(t) \lesssim (1+t)^{2M}. \tag{64}$$

for some $N = N(t)$. (See (71), (72) below for the definition of N and the growth rate M we eventually establish). The following proposition, represents an "almost conservation law" and will yield (64).

Proposition 5.6. *Given $s > \frac{4}{7}, N \gg 1$, and initial data $u_0 \in C_0^\infty(\mathbb{R}^2)$ (see preceeding remark) with $E(I_N\phi_0) \leq 1$, then there exists a $\delta = \delta(||u_0||_{L^2(\mathbb{R}^2)}) > 0$ so that the solution*

$$u(x,t) \in C([0,\delta], H^s(\mathbb{R}^2))$$

of (54) satisfies

$$E(I_N u)(t) = E(I_N u)(0) + O(N^{-\frac{3}{2}+}), \tag{65}$$

for all $t \in [0,\delta]$.

We first show that Proposition 5.6 implies (64). Recall that the initial value problem here has a scaling symmetry, and is H^s-subcritical when $1 > s > 0$, and $n = 2$. That is, if u is a solution, so too

$$u_\lambda(x,t) := \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right). \tag{66}$$

Using (61), the following energy can be made arbitrarily small by taking λ large,

$$E(I_N u_{\lambda,0}) \leq \left((N^{2-2s})\lambda^{-2s} + \lambda^{-2}\right) \cdot (1 + ||u_0||_{H^s(\mathbb{R}^2)})^4 \tag{67}$$

$$\leq C_0(N^{2-2s}\lambda^{-2s}) \cdot (1 + ||u_0||_{H^s(\mathbb{R}^2)})^4. \tag{68}$$

It is important to remark that since the problem is L^2 critical, $\|u_0\|_{L^2} \sim \|u_{\lambda,0}\|_{L^2}$. Assuming $N \gg 1$ is given[18], we choose our scaling parameter $\lambda = \lambda(N, ||u_0||_{H^s(\mathbb{R}^2)})$

$$\lambda = N^{\frac{1-s}{s}} \left(\frac{1}{2C_0}\right)^{-\frac{1}{2s}} \cdot \left(1 + ||u_0||_{H^s(\mathbb{R}^2)}\right)^{\frac{2}{s}} \tag{69}$$

so that $E(I_N u_{\lambda,0}) \leq \frac{1}{2}$. We may now apply Proposition 5.6 to the scaled initial data $u_{\lambda,0}$, and in fact we may reapply this proposition until the size of $E(I_N u_\lambda)(t)$ reaches 1, that is at least $C_1 \cdot N^{\frac{3}{2}-}$ times. Hence

$$E(I_N u_\lambda)(C_1 N^{\frac{3}{2}-}\delta) \sim 1. \tag{70}$$

We now have to undo the scaling: given any $T_0 \gg 1$, we establish the polynomial growth (64) from (70) by first choosing our parameter $N \gg 1$ so that

$$T_0 \sim \frac{N^{\frac{3}{2}-}}{\lambda^2} C_1 \cdot \delta \sim N^{\frac{7s-4}{2s}-}, \tag{71}$$

where we've kept in mind (69). Note the exponent of N on the right of (71) is positive provided $s > \frac{4}{7}$, hence the definition of N makes sense for arbitrary T_0. In two space dimensions,

$$E(I_N u)(t) = \lambda^2 E(I_N u_\lambda)(\lambda^2 t).$$

We use (69), (70), and (71) to conclude that for $T_0 \gg 1$,

$$E(I_N u)(T_0) \leq C_2 T_0^{\frac{1-s}{\frac{7}{4}s-1}+}, \tag{72}$$

where N is chosen as in (71) and $C_2 = C_2(||u_0||_{H^s(\mathbb{R}^2)}, \delta)$. Together with (62), the bound (72) establishes the desired polynomial bound (63).

It remains then to prove Proposition 5.6. We will need the following modified version of the usual local existence theorem, wherein we control for small times the smoothed solution in the $X^\delta_{1,\frac{1}{2}+}$ norm.

Proposition 5.7. *Assume $\frac{4}{7} < s < 1$ and we are given data for the IVP (54) with $E(Iu_0) \leq 1$. Then there is a constant $\delta = \delta(||u_0||_{L^2(\mathbb{R}^2)})$ so that the solution u obeys the following bound on the time interval $[0, \delta]$,*

$$||Iu||_{X^\delta_{1,\frac{1}{2}+}} \lesssim 1. \tag{73}$$

[18]The parameter N will be chosen shortly.

Proof. We mimic the typical iteration argument showing local existence. We will need the following three estimates involving the $X_{s,\delta}$ spaces (55) and functions $F(x,t)$, $f(x)$. (Throughout this section, the implicit constants in the notation $\lesssim$ are independent of δ.)

$$\|S(t)f\|_{X^{\delta}_{1,\frac{1}{2}+}} \lesssim \|f\|_{H^1(\mathbb{R}^2)}, \tag{74}$$

$$\left\| \int_0^t S(t-\tau)F(x,\tau)d\tau \right\|_{X_{1,\frac{1}{2}+}} \lesssim \|F\|_{X^{\delta}_{1,-\frac{1}{2}+}}, \tag{75}$$

$$\|F\|_{X^{\delta}_{1,-b}} \lesssim \delta^P \|F\|_{X^{\delta}_{1,-\beta}}, \tag{76}$$

where in (76) we have $0 < \beta < b < \frac{1}{2}$, and $P = \frac{1}{2}(1 - \frac{\beta}{b}) > 0$. The bounds (74), (75) are analogous to estimates (3.13), (3.15) in [54]. As for (76), by duality it suffices to show

$$||F||_{X^{\delta}_{-1,\beta}} \lesssim \delta^P ||F||_{X^{\delta}_{-1,b}}.$$

Interpolation gives

$$||F||_{X^{\delta}_{-1,\beta}} \lesssim ||F||^{(1-\frac{\beta}{b})-}_{X^{\delta}_{-1,0}} \cdot ||F||^{\frac{\beta}{b}}_{X^{\delta}_{-1,b}}.$$

As $b \in (0, \frac{1}{2})$, arguing exactly as on page 771 of [33],

$$||F||_{X^{\delta}_{-1,0}} \lesssim \delta^{\frac{1}{2}} ||F||_{X^{\delta}_{-1,b}},$$

and (76) follows.

Duhamel's principle gives us

$$\begin{aligned} ||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} &= \left\| S(t)(Iu_0) + \int_0^t S(t-\tau) I(u\bar{u}u)(\tau)d\tau \right\|_{X^{\delta}_{1,\frac{1}{2}+}} \\ &\lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + ||I(u\overline{u}u)||_{X^{\delta}_{1,-\frac{1}{2}+}} \\ &\lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon} ||I(u\overline{u}u)||_{X^{\delta}_{1,-\frac{1}{2}++}}, \end{aligned} \tag{77}$$

where $-\frac{1}{2}++$ is a real number slightly larger than $-\frac{1}{2}+$ and $\epsilon > 0$. By the definition of the restricted norm (56),

$$||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} \lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon} ||I(\psi\overline{\psi}\psi)||_{X_{1,-\frac{1}{2}++}}, \tag{78}$$

where the function ψ agrees with u for $t \in [0, \delta]$, and

$$||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} \sim \quad ||I\psi||_{X_{1,\frac{1}{2}+}}. \tag{79}$$

We will show shortly that

$$||I(\psi\overline{\psi}\psi)||_{X_{1,-\frac{1}{2}++}} \lesssim \quad ||I\psi||^3_{X_{1,\frac{1}{2}+}}. \tag{80}$$

Setting then $Q(\delta) \equiv ||Iu(t)||_{X^{\delta}_{1,\frac{1}{2}+}}$, the bounds (77), (79) and (80) yield

$$Q(\delta) \lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon}(Q(\delta))^3. \tag{81}$$

Note

$$||Iu_0||_{H^1(\mathbb{R}^2)} \lesssim (E(Iu_0))^{\frac{1}{2}} + ||u_0||_{L^2(\mathbb{R}^2)} \lesssim 1 + ||u_0||_{L^2(\mathbb{R}^2)}. \tag{82}$$

As Q is continuous in the variable δ, a bootstrap argument yields (73) from (81), (82).

It remains to show (80). Using the interpolation lemma of [31], it suffices to show

$$||\psi\bar{\psi}\psi||_{X_{s,-\frac{1}{2}++}} \lesssim ||\psi||^3_{X_{s,\frac{1}{2}+}}, \tag{83}$$

for all $\frac{4}{7} < s < 1$. By duality and a "Leibniz" rule[19], (83) follows from

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\langle \nabla \rangle^s u_1) \overline{u_2} u_3 u_4 dx dt \right| \lesssim ||u_1||_{X_{s,\frac{1}{2}+}} \cdot ||u_2||_{X_{s,\frac{1}{2}+}} \cdot ||u_3||_{X_{s,\frac{1}{2}+}} ||u_4||_{X_{0,\frac{1}{2}--}}. \tag{84}$$

Note that since the factors in the integrand on the left here will be taken in absolute value, the relative placement of complex conjugates is irrelevant. Use Hölder's inequality on the left side of (84), taking the factors in, respectively, $L^4_{x,t}, L^4_{x,t}, L^6_{x,t}$ and $L^3_{x,t}$. Using a Strichartz inequality,

$$\begin{aligned} ||\langle \nabla \rangle^s u_1||_{L^4_{x,t}(\mathbb{R}^{2+1})} &\lesssim ||\langle \nabla \rangle^s u_1||_{X_{0,\frac{1}{2}+}} \\ &= ||u_1||_{X_{s,\frac{1}{2}+}}, \end{aligned}$$

and

$$\begin{aligned} ||u_2||_{L^4_{x,t}(\mathbb{R}^{2+1})} &\lesssim ||u_2||_{X_{0,\frac{1}{2}+}} \\ &\lesssim ||u_2||_{X_{s,\frac{1}{2}+}}. \end{aligned}$$

The bound for the third factor uses Sobolev embedding and the $L^6_t L^3_x$ Strichartz estimate,

$$\begin{aligned} ||u_3||_{L^6_t L^6_x(\mathbb{R}^{2+1})} &\lesssim ||\langle \nabla \rangle^{\frac{1}{3}} u_3||_{L^6_t L^3_x(\mathbb{R}^{2+1})} \\ &\lesssim ||\langle \nabla \rangle^{\frac{1}{3}} u_3||_{X_{0,\frac{1}{2}+}} \\ &\leq ||u_3||_{X_{s,\frac{1}{2}+}}. \end{aligned}$$

It remains to bound $||u_4||_{L^3(\mathbb{R}^{2+1})}$. Interpolating between $||u_4||_{L^2_t L^2_x} \leq ||u_4||_{X_{0,0}}$ and the Strichartz estimate $||u_4||_{L^4_t L^4_x} \lesssim ||u_4||_{X_{0,\frac{1}{2}+}}$ yields

$$||u_4||_{L^3_t L^3_x} \lesssim ||u_4||_{X_{0,\frac{1}{2}--}}.$$

This completes the proof of (84), and hence Proposition 5.7. □

Before we proceed to the proof of Proposition 5.6 we would like to present the proof of conservation of mass[20] for (54) using Fourier transform. Understanding this proof is fundamental to understand the types of cancelations that will make $E(Iu)$ almost conserved.

Proposition 5.8. *Assume that u is a solution to* (54) *smooth and decaying at infinity. Then* $\|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2}$.

Proof. We write this L^2 norm using Plancherel formula

$$\|u(t)\|^2_{L^2} = \int \hat{u}(\xi,t)\overline{\hat{u}}(\xi,t)\, d\xi$$

[19] By this, we mean the operator $\langle D \rangle^s$ can be distributed over the product by taking Fourier transform and using $\langle \xi_1 + \dots \xi_4 \rangle^s \lesssim \langle \xi_1 \rangle^s + \dots \langle \xi_4 \rangle^s$.

[20] Actually showing the proof of conservation of energy would be even more appropriate here since in Proposition 5.6 we will be dealing with an energy instead of a mass, but clearly for the mass the calculation is less involved and the ideas are still present in full power!

Using the equation we then have

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2Re\int (\hat{u}(\xi,t))_t\bar{\hat{u}}(\xi,t)\,d\xi \\
&= -Im\int |\xi|^2\hat{u}(\xi,t)\bar{\hat{u}}(\xi,t)\,d\xi - 2Im\int \widehat{u^2\bar{u}}(\xi)\bar{\hat{u}}(\xi,t)\,d\xi \\
&= -2Im\int_{\xi_1+\xi_2+\xi_3-\xi=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(\xi)\,d\xi d\xi_1 d\xi_2 d\xi_3 \\
&= -2Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4
\end{aligned}
$$

and by symmetry

$$
\begin{aligned}
& 2Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 = \\
& Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
+\; & Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(-\xi_2)\bar{\hat{u}}(\xi_1)\hat{u}(-\xi_4)\bar{\hat{u}}(\xi_3)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 = 0
\end{aligned}
$$

□

Problem 5.9. *Prove the conservation of energy* (23) *by using Fourier transform.*

Proof of Proposition 5.6. The usual energy (23) is shown to be conserved by differentiating in time, integrating by parts, and using the equation (54),

$$
\begin{aligned}
\partial_t E(u) &= \mathrm{Re}\int_{\mathbb{R}^2} \overline{u_t}(|u|^2u - \Delta u)dx \\
&= \mathrm{Re}\int_{\mathbb{R}^2} \overline{u_t}(|u|^2u - \Delta u - iu_t)dx \\
&= 0.
\end{aligned}
$$

We follow the same strategy to estimate the growth of $E(Iu)(t)$,

$$
\begin{aligned}
\partial_t E(Iu)(t) &= \mathrm{Re}\int_{\mathbb{R}^2} \overline{I(u)_t}(|Iu|^2Iu - \Delta Iu - iIu_t)dx \\
&= \mathrm{Re}\int_{\mathbb{R}^2} \overline{I(u)_t}(|Iu|^2Iu - I(|u|^2u))dx,
\end{aligned}
$$

where in the last step we've applied I to (54). When we integrate in time and apply the Parseval formula[21] it remains for us to bound

$$
\begin{aligned}
(85)\quad & E(Iu(\delta)) - E(Iu(0)) = \\
& \int_0^\delta \int_{\sum_{j=1}^4 \xi_j=0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{I\partial_t u}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4).
\end{aligned}
$$

The reader may ignore the appearance of complex conjugates here and in the sequel, as they have no impact on the availability of estimates, (see e.g. Lemma 5.4 above). We include the complex conjugates for completeness.

[21]That is, $\int_{\mathbb{R}^n} f_1(x)f_2(x)f_3(x)f_4(x)dx = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)$ where $\int_{\sum_i \xi_i=0}$ here denotes integration with respect to the hyperplane's measure
$\delta_0(\xi_1+\xi_2+\xi_3+\xi_4)d\xi_1 d\xi_2 d\xi_3 d\xi_4$, with δ_0 the one dimensional Dirac mass.

We use the equation to substitute for $\partial_t I(u)$ in (85). Our aim is to show that

$$\text{Term}_1 + \text{Term}_2 \lesssim N^{-\frac{3}{2}+}, \tag{86}$$

where the two terms on the left are

$$\text{Term}_1 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \widehat{(\Delta \overline{Iu})}(\xi_1) \cdot \widehat{Iu}(\xi_2) \cdot \widehat{\overline{Iu}}(\xi_3) \cdot \widehat{Iu}(\xi_4) \right| \tag{87}$$

$$\text{Term}_2 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \widehat{(\overline{I(|u|^2 u)})}(\xi_1) \cdot \widehat{Iu}(\xi_2) \cdot \widehat{\overline{Iu}}(\xi_3) \cdot \widehat{Iu}(\xi_4) \right|. \tag{88}$$

From this point on the proof proceeds with a case by case analysis based on the relative magnitude of various frequencies. The basic cancellation of the type we presented in the proof of Proposition 5.8 are fundamental as is the fact that the multiplier is smooth. We send the reader to the original paper for a complete proof.

□

Remark 5.10. Here we only gave an idea of the "I-method". One can implement it in more effective ways by defining formally families of energies that, if controlled analytically, are proved to be more and more *almost conserved*. This was in fact the case for the one dimensional derivative NLS [24, 25] and the KdV [27] for example. Unfortunately controlling these families of energies becomes more difficult in higher dimensions since orthogonality issues start appearing, see for example [30].

6. Lecture # 5: Interaction Morawetz estimates and scattering

In the last lecture we discussed the question of global well-posedness. Once one can prove that given an initial data a unique solution evolving from that data exists for all times it becomes natural to ask how this solution looks like as $t \to \pm\infty$. The theory that addresses these questions is called *scattering theory*. In order to put scattering in a more general context we need few definitions. We will give them by assuming that the solution for (1) is defined globally in time with respect to the energy space H^1, but it will be easy to generalize them when more general Sobolev spaces are considered.

Definition 6.1 (Scattering)**.** Given a global solution $u \in H^1$ to (1) we say that u *scatters* to $u_+ \in H^1$ if

$$\|u(t) - S(t)u_+\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty. \tag{89}$$

Clearly a similar definition is given if $t \to -\infty$.

Remark 6.2. Using the properties of the group $S(t)$ it is easy to see that (89) is equivalent to

$$\|S(-t)u(t) - u_+\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty. \tag{90}$$

Since by the Duhamel formula (6)

$$S(-t)u(t) - u_+ = u_0 - u_+ - i\int_0^t S(-t')|u(t')|^{p-1}u(t')\,dt'$$

it is clear that scattering is **equivalent** to showing that the improper time integral

$$\int_0^\infty S(-t')|u(t')|^{p-1}u(t')\,dt'$$

converges in H^1 and in particular this will give the formula for u_+, i.e.

$$u_+ = u_0 - i\int_0^\infty S(-t')|u(t')|^{p-1}u(t')\,dt'. \tag{91}$$

One can also consider an inverse problem: assume $u_+ \in H^1$, can we find an initial data $u_0 \in H^1$ such that the global solution u for (1) scatters to u_+?

Definition 6.3 (Wave Operator)**.** Assume that for any $u_+ \in H^1$ there exists $u_0 \in H^1$ such that the solution u to (1) scatters to u_+ in the sense of (91). Then we define the wave operator

$$\Omega^+ : H^1 \longrightarrow H^1 \quad \text{such that} \quad \Omega^+(u_+) = u_0$$

.

In order to prove the existence of Ω_+ it is useful to write the solution u in terms of u_+. In fact using the Duhamel representation (6) and (91) above we can write

$$u(t) = S(t)u_+ + i\int_t^\infty S(t-t')(|u(t')|^{p-1}u(t')\,dt', \tag{92}$$

and being able to define Ω_+ is equivalent to being able to define (92) for $t = 0$.

Remark 6.4. From the two definitions given above it is clear that proving scattering is equivalent to proving that the wave operator Ω^+ is invertible. In this case we also say that we have *Asymptotic Completeness.*

At first, from the definitions, it is not clear what is harder to prove, if existence of the wave operator or asymptotic completeness. But in practice the former is easier. One of the reasons is that the existence of the wave operator usually follows from the strong[22] dispersive estimates (10) and from iteration of local well-posedness. On the other hand to prove scattering one needs global space time bounds that are very difficult to get. Here we only address the question of existence of the wave operator (see [19]) briefly in Theorem 6.14, but we will concentrate on the scattering issue much more. The bibliography on scattering is quite large (see for example [19] for a good list of results), but certainly the work of Ginibre and Velo (see for example [40]) takes a special stand in it. But in this lecture we will take a different and more recent approach that is based on the so called *Interaction Morawetz Estimates* [28, 71, 76].

6.5. **Interaction Morawetz Estimates.** At this point there are several ways one can present these estimates: as weighted overages of the classical Morawetz estimates presented in Lecture #2 [28, 76], as classical Morawetz estimates applied to tensors of solutions to (1) [20, 42, 43], or as more general and refined calculations dealing with vector fields [32, 66]. Here we describe the first one, which was also the original one given in 3 dimensions[23].

In the following we introduce an interaction potential generalization of the classical Morawetz action and associated inequalities. We first recall the standard Morawetz action centered at a point and the proof that this action is monotonically increasing with time when the nonlinearity is defocusing. The interaction generalization is introduced in the second subsection. The key consequence of the analysis in this section is the $L^4_{x,t}$ estimate (116).

The discussion in this section will be carried out in the context of the following generalization of (1):

$$i\partial_t u + \alpha \Delta u = \mu f(|u|^2)u, \qquad u : \mathbb{R} \times \mathbb{R}^3 \longmapsto \mathbb{C}, \tag{93}$$

$$u(0) = u_0. \tag{94}$$

Here f is a smooth function $f : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ and α and μ are real constants that permit us to easily distinguish in the analysis below those terms arising from the Laplacian or the nonlinearity. We also define $F(z) = \int_0^z f(s)ds$.

We will use polar coordinates $x = r\omega,\ r > 0, \omega \in S^2$, and write Δ_ω for the Laplace-Beltrami operator on S^2. For ease of reference below, we record some alternate forms of the equation in (93):

$$u_t = i\alpha\Delta u - i\mu f(|u|^2)u, \tag{95}$$

$$\overline{u}_t = -i\alpha\Delta\overline{u} + i\mu f(|u|^2)\overline{u}, \tag{96}$$

$$u_t = i\alpha u_{rr} + i\frac{2\alpha}{r}u_r + i\frac{\alpha}{r^2}\Delta_\omega u - i\mu f(|u|^2)u, \tag{97}$$

$$(ru_t) = i\alpha(ru)_{rr} + i\frac{\alpha}{r}\Delta_\omega u - i\mu r f(|u|^2)u, \tag{98}$$

$$(r\overline{u}_t) = -i\alpha(r\overline{u})_{rr} - i\frac{\alpha}{r}\Delta_\omega \overline{u} + i\mu f(|u|^2)\overline{u}. \tag{99}$$

[22] Especially in higher dimensions.

[23] The reader will see below that for $n = 1, 2$ the argument breaks down. In fact for $n = 1$ one needs to use tensors of solutions [20] and for $n = 2$ one either is happy with a local in time estimate [37] or needs to introduce a much more refined argument [32]. For $n > 3$ the argument below can be used but the estimates are less "clean" than the $L^4_t L^4_x$ norm we find below. But some use of standard harmonic analysis leads to a better space time estimates which is as good as the one we prove here [68, 75, 76].

6.6. **Standard Morawetz action and inequalities.** We will call the following quantity the *Morawetz action centered at 0* for the solution u of (93) and this should be compared with (29),

$$M_0[u](t) = \int_{\mathbb{R}^3} \mathrm{Im}[\bar{u}(t,x)\nabla u(t,x)] \cdot \frac{x}{|x|} dx. \tag{100}$$

We check using the equation that,

$$\partial_t(|u|^2) = -2\alpha \nabla \cdot \mathrm{Im}[\overline{u}(t,x)\nabla u(t,x)], \tag{101}$$

hence we may interpret M_0 as the spatial average of the radial component of the L^2-mass current. We might expect that M_0 will increase with time if the wave u scatters since such behavior involves a broadening redistribution of the L^2-mass. The following proposition of Lin and Strauss [59] that is equivalent to (29), indeed gives $\frac{d}{dt}M_0[u](t) \geq 0$ for defocusing equations.

Proposition 6.7. [59] *If u solves* (93)-(94) *then the Morawetz action at 0 satisfies the identity*

$$\partial_t M_0[u](t) = 4\pi\alpha |u(t,0)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x|} |\nabla\!\!\!/_0 u(t,x)|^2 dx + \mu \int_{\mathbb{R}^3} \frac{2}{|x|} \left\{ |u|^2 f(|u|^2)(t) - F(|u|^2) \right\} dx. \tag{102}$$

where $\nabla\!\!\!/_0$ is the angular component of the derivative,

$$\nabla\!\!\!/_0 u = \nabla u - \frac{x}{|x|}(\frac{x}{|x|} \cdot \nabla u). \tag{103}$$

In particular, M_0 is an increasing function of time if the equation (93) *satisfies the repulsivity condition,*

$$\mu \left\{ |u|^2 f(|u|^2)(t) - F(|u|^2) \right\} \geq 0. \tag{104}$$

Note that for pure power potentials $F(x) = \frac{2}{p+1} x^{\frac{p+1}{2}}$, where the nonlinear term in (93) is $|u|^{p-1}u$, the function $|u|^2 f(|u|^2) - F(|u|^2) = \frac{p-1}{2} F(|u|^2)$. Hence condition (104) holds.

We may center the above argument at any other point $y \in \mathbb{R}^3$ with corresponding results. Toward this end, define the *Morawetz action centered at y* to be,

$$M_y[u](t) = \int_{\mathbb{R}^3} \mathrm{Im}[\overline{u}(x)\nabla u(x)] \cdot \frac{x-y}{|x-y|} dx. \tag{105}$$

We shall often drop the u from this notation, as we did previously in writing $M_0(t)$.

Corollary 6.8. *If u solves* (93) *the Morawetz action at y satisfies the identity*

$$\frac{d}{dt} M_y = 4\pi\alpha |u(t,y)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla\!\!\!/_y u(t,x)|^2 dx + \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \left\{ |u|^2 f(|u|^2) - F(|u|^2) \right\} dx, \tag{106}$$

where $\nabla\!\!\!/_y u \equiv \nabla u - \frac{x-y}{|x-y|} \left(\frac{x-y}{|x-y|} \cdot \nabla u \right)$. In particular, M_y is an increasing function of time if the nonlinearity satisfies the repulsivity condition (104).

Corollary 6.8 shows that a solution is, on average, repulsed from any fixed point y in the sense that $M_y[u](t)$ is increasing with time.

For our scattering results, we'll need the following pointwise bound for $M_y[u](t)$.

Lemma 6.9. *Assume u is a solution of* (93) *and $M_y[u](t)$ as in* (105). *Then,*

$$|M_y[u](t)| \lesssim \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}_x}. \tag{107}$$

Proof. Without loss of generality we take $y = 0$. This is a refinement of the easy bound using Cauchy-Schwarz $|M_y[u](t)| \lesssim \|u(t)\|_{L^2_x} \|\nabla u(t)\|_{L^2_x}$. By duality

$$|\operatorname{Im} \int_{\mathbb{R}^3} \overline{u(x,t)} \partial_r u(x,t) dx \,| \leq \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \cdot \|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}.$$

It suffices to show $\|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \leq \|u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$. By duality and the definition $\partial_r \equiv \frac{x}{|x|} \cdot \nabla$, it remains to prove,

$$\| \frac{x}{|x|} f \|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \tag{108}$$

for any f for which the right hand side is finite. Inequality (108) follows from interpolating between the following two bounds,

$$\| \frac{x}{|x|} f \|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}$$
$$\| \frac{x}{|x|} f \|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|f\|_{H^1(\mathbb{R}^3)}$$

the first of which is trivial, the second of which follows from Hardy's inequality,

$$\begin{aligned} \|\nabla \left(\frac{x}{|x|} f \right) \|_{L^2} &\leq \| \frac{x}{|x|} \cdot \nabla f \|_{L^2} + \| \frac{1}{|x|} f \|_{L^2} \\ &\lesssim \|\nabla f\|_{L^2}. \end{aligned}$$

□

The well-known Morawetz-type inequalities, so useful in proving local decay or scattering for (93), arise by integrating the identity (102) or (106) in time. For nonlinear Schrödinger equations, this argument appears in the work of Lin and Strauss [59], who cite as motivation earlier work on Klein-Gordon equations by Morawetz [62].

Corollary 6.10 (Morawetz estimate centered at y.)**.** *Suppose u solves* (93)-(94)*. Then for any $y \in \mathbb{R}^3$,*

$$\begin{aligned} 2 \sup_{t \in [0,T]} \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}_x} \gtrsim 4\pi\alpha \int_0^T |u(t,y)|^2 dt + \int_0^T \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla\!\!\!/_y u(t,x)|^2 dx dt \\ + \int_0^T \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \left\{ |u|^2 f(|u|^2) - F(|u|^2) \right\} dx dt. \end{aligned} \tag{109}$$

Assuming (93) has a repulsive nonlinearity as in (104), all terms on the right side of the inequality (109) are positive. The inequality therefore gives in particular a bound uniform in T for the quantity $\int_0^T \int_{\mathbb{R}^3} \frac{|u(t,x)|^4}{|x-y|} dx dt$, for solutions u of the defocusing (1), when $p = 3$.

In their proof of scattering in the energy space for the cubic defocusing problem (1), Ginibre and Velo [40] combine this relatively localized[24] decay estimate with a bound surrogate for finite propagation speed in order to show the solution is in certain global-in-time Lebesgue spaces $L^q([0,\infty), L^r(\mathbb{R}^3))$. Scattering follows rather quickly, as will be shown later.

In the following section, we show how to establish an unweighted, global in time Lebesgue space bound directly. The argument below involves the identity (106), but our estimate arises eventually from the linear part of the equation, more specifically from the first term on the right of (106), rather than the third (nonlinearity) term.

[24]The bound mentioned here may be considered localized since it implies decay of the solution near the fixed point y, but doesn't preclude the solution staying large at a point which moves rapidly away from y, for example.

6.11. **Morawetz interaction potential.** Given a solution u of (93), we define the *Morawetz interaction potential* to be

$$M(t) = \int_{\mathbb{R}^3} |u(t,y)|^2 M_y(t)dy. \tag{110}$$

The bound (107) immediately implies

$$|M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2. \tag{111}$$

If u solves (93) then the identity (106) gives us the following identity for $\frac{d}{dt}M(t)$,

$$\begin{aligned} \frac{d}{dt}M(t) \;=\; & 4\pi\alpha \int_y |u(y)|^4 dy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|}|u(y)|^2|\nabla\!\!\!\!/_y u(x)|^2 dxdy \\ & + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\mu}{|x-y|}|u(y)|^2 \left\{|u(x)|^2 f(|u(x)|^2) - F(|u(x)|^2)\right\} dxdy \\ & + \int_{\mathbb{R}^3} \partial_t(|u(t,y)|^2)\; M_y(t)dy. \end{aligned} \tag{112}$$

We write the right side of (112) as $I + II + III + IV$, and work now to rewrite this as a sum involving nonnegative terms.

Proposition 6.12. *Referring to the terms comprising* (112), *we have*

$$IV \geq -II. \tag{113}$$

Consequently, solutions of (93) *satisfy*

$$\frac{d}{dt}M(t) \geq 4\pi\alpha \int_{\mathbb{R}^3} |u(t,y)|^4 dy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\mu}{|x-y|}|u(t,y)|^2 \left\{|u|^2 f(|u|^2) - F(|u|^2)\right\} dxdy. \tag{114}$$

In particular, $M(t)$ is monotone increasing for equations with repulsive nonlinearities.

Assuming Proposition 6.12 for the moment, we combine (111) and (114) to obtain the following estimate which plays the major new role in our scattering analysis below,

Corollary 6.13. *Take u to be a smooth solution to the initial value problem* (93)-(94) *above, under the repulsivity assumption* (104). *Then we have the following interaction Morawetz inequalities,*

$$\begin{aligned} 2\|u(0)\|_{L^2}^2 \sup_{t\in[0,T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 \;\gtrsim\; & 4\pi\alpha \int_0^T \int_{\mathbb{R}^3} |u(t,y)|^4 dydt \\ & + \int_0^T \int_y \int_x \frac{2\mu}{|x-y|}|u(t,y)|^2 \left\{|u|^2 f(|u|^2) - F(|u|^2)\right\}(t,x)dxdydt. \end{aligned} \tag{115}$$

In particular, we obtain the following spacetime $L^4([0,T]\times\mathbb{R}^3)$ estimate,

$$\int_0^T \int_{\mathbb{R}^3} |u(t,y)|^4 dydt \leq C\|u_0\|_{L^2(\mathbb{R}^3)}^2 \sup_{t\in[0,T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2, \tag{116}$$

where C is independent of T.

Of course, for solutions of the defocusing IVP (1) starting from finite energy initial data, the right side of (116) is uniformly bounded by energy considerations - leading to a rather direct proof of the result in [40] of scattering in the energy space that we will present below.

Proof. We now turn to the proof of Proposition 6.12. Use (101) to write

$$\begin{aligned} IV &= -\int_{\mathbb{R}^3_y} \nabla \cdot \mathrm{Im}[2\alpha \overline{u}(y) \nabla u(y)] M_y(t) dy \\ &= -\int_y \int_x \partial_{y_l} \mathrm{Im}[2\alpha \overline{u}(y) \partial_{y_l} u(y)]\ \mathrm{Im}[\overline{u}(x) \frac{x_m - y_m}{|x-y|} \partial_{x_m} u(x)] dx dy, \end{aligned}$$

where repeated indices are implicitly summed. We integrate by parts in y, moving the leading ∂_{y_l} to the unit vector $\frac{x-y}{|x-y|}$. Note that,

$$\partial_{y_l} \left(\frac{x_m - y_m}{|x-y|} \right) = \frac{-\delta_{lm}}{|x-y|} + \frac{(x_l - y_l)(x_m - y_m)}{|x-y|^3}. \tag{117}$$

Write $\mathbf{p}(x) = \mathrm{Im}[\overline{u}(x) \nabla u(x)]$ for the mass current at x and use (117) to obtain

$$IV = -2\alpha \int_y \int_x \left[\mathbf{p}(y) \cdot \mathbf{p}(x) - (\mathbf{p}(y) \cdot \frac{x-y}{|x-y|})(\mathbf{p}(x) \cdot \frac{x-y}{|x-y|}) \right] \frac{dx dy}{|x-y|}. \tag{118}$$

The preceding integrand has a natural geometric interpretation. We are removing the inner product of the components of $\mathbf{p}(y)$ and $\mathbf{p}(x)$ parallel to the vector $\frac{x-y}{|x-y|}$ from the full inner product of $\mathbf{p}(y)$ and $\mathbf{p}(x)$. This amounts to taking the inner product of $\pi_{(x-y)^\perp} \mathbf{p}(y) \cdot \pi_{(x-y)^\perp} \mathbf{p}(x)$ where we have introduced the projections onto the subspace of $\mathbb{R}^3$ perpendicular to the vector $\frac{x-y}{|x-y|}$. But

$$|\pi_{(x-y)^\perp} \mathbf{p}(y)| = \left| \mathbf{p}(y) - \frac{x-y}{|x-y|} \big(\frac{x-y}{|x-y|} \cdot \mathbf{p}(y) \big) \right| = |\mathrm{Im}[\overline{u}(y) \nabla\!\!\!/_x u(y)| \leq |u(y)| \cdot |\nabla\!\!\!/_x u(y)|. \tag{119}$$

A similar identity and inequality holds upon switching the roles of x and y in (119). We have thus shown that

$$IV \geq -2\alpha \int_y \int_x |u(x)| \cdot |\nabla\!\!\!/_y u(x)| \cdot |u(y)| \cdot |\nabla\!\!\!/_x u(y)| \frac{dx dy}{|x-y|}. \tag{120}$$

The conclusion (113) follows by applying the elementary bound $|ab| \leq \frac{1}{2}(a^2 + b^2)$ with $a = |u(y)| \cdot |\nabla\!\!\!/_y u(x)|$ and $b = |u(x)| \cdot |\nabla\!\!\!/_x u(y)|$. ☐

We now state the following theorem as an example of how to use Morawetz interaction estimates in order to prove scattering

Theorem 6.14. *Consider the cubic, defocusing, NLS* (1) *in* $\mathbb{R}^3$. *Then the wave operator exists and there is asymptotic completeness.*

Remark 6.15. Theorem 6.14 is not the best known result for this cubic NLS. In fact in [28] this same IVP was considered and the $L^4_t L^4_x$ Morawetz estimate was used to prove scattering below H^1. For other H^1 subcritical scattering results one should also consult [76] when $n \geq 3$, [32] when $n = 2$ and [20] when $n = 1$. In these cases if one wants to show scattering with regularity $s < 1$, for example when $n = 3$ in [28], the argument is more complicated than the one described for H^1 since one has to prove that the H^s norm of the solution is bounded by using the "I-method" as in Lecture # 4. The basic idea though is the same.

Proof. **Existence of Ω_+:** we go back to the formula (92). The idea is to go first from $t = +\infty$ to $t = T$ for some $T > 0$ using some smallness and then solve the problem in the finite interval of time backward from T to 0.

We know already in what kind of spaces we can argue by contraction method: the space S^1 containing all the admissible Strichartz norms of the function and its derivatives and possibly also those that are embedded into these norms by the Sobolev theorem. But in this case there is one more request that we want to make. We want a smallness assumption, possibly obtained by shrinking the time interval or better by taking the time interval at infinity where the "tail" of the function lives. For this reason we should avoid any norm that contains a L^∞_t. So we proceed in two steps first we consider the smaller space $\tilde{S}^1$ given by the norm

$$\|f\|_{\tilde{S}^1} = \|f\|_{L^5_t L^5_x} + \|f\|_{L^{10/3}_t W^{1,10/3}_x}.$$

Notice that by Sobolev

$$\|f\|_{L^5_t L^5_x} \lesssim \|f\|_{L^5_t W^{1,30/11}_x}$$

and $(5, 30/11)$ is a Strichartz admissible pair. It follows that if $u_+ \in H^1$ then by (12)

$$\|S(t)u_+\|_{\tilde{S}^1_{[T,\infty)}} \leq \epsilon \tag{121}$$

for T large enough. From (92) if we define

$$Lv(t) = S(t)u_+ + i\int_t^\infty S(t-t')(|v(t')|^2 v(t')\, dt', \tag{122}$$

and we use (13), where we pick the couple $(\tilde{q}, \tilde{r}) = (10/3, 10/3)$, we have
(123)

$$\|Lv\|_{\tilde{S}^1_{[T,\infty)}} \leq \epsilon + C\||v|^2\nabla v\|_{L^{10/7}_{[T,\infty)} L^{10/7}_x} \leq \epsilon + C\|v\|^2_{L^5_{[T,\infty)} L^5_x}\|v\|_{L^{10/3}_{[T,\infty)} W^{1,10/3}_x} = \epsilon + C\|v\|^3_{\tilde{S}^1_{[T,\infty)}}.$$

and with a similar estimate

$$\|Lv - Lw\|_{\tilde{S}^1_{[T,\infty)}} \leq C(\|v\|_{\tilde{S}^1_{[T,\infty)}} + \|w\|_{\tilde{S}^1_{[T,\infty)}})\|v-w\|_{\tilde{S}^1_{[T,\infty)}}. \tag{124}$$

and thanks to the presence of ϵ one can proceed with the contraction argument. This would give a solution in $[T, \infty)$, which in particular has the property that

$$\|u\|_{\tilde{S}^1_{[T,\infty)}} \lesssim \epsilon \tag{125}$$

But we didn't prove that this solution is in $C([T,\infty), H^1)$ for example. To do this we need to go back and estimate the solution u in the Strichartz space $S^1_{[T,\infty)}$. We in fact have by (12) and (13)

$$\|u\|_{S^1} \leq C\|u_+\|_{H^1} + C\||u|^2\nabla u\|_{L^{10/7}_{[T,\infty)} L^{10/7}_x}$$

and from (125)

$$\|u\|_{S^1} \leq C\|u_+\|_{H^1} + C\|u\|^3_{\tilde{S}^1_{[T,\infty)}} \lesssim \|u_+\|_{H^1},$$

and we are done in the interval $[T, \infty)$.

We now need to proceed from $t = T$ back to $t = 0$. Since the problem is subcritical, an iteration of local well-posedness like we presented in Lecture # 4, using the conservation of the energy and mass, will suffice to cover the finite interval $[0, T]$.

Invertibility of Ω_+: This is the proof of scattering and we need to go back to (91). From here we see that we only need to show that the integral involving the global solution u

$$\int_0^\infty S(t)(|u|^2 u)(t)\, dt$$

converges in H^1. By the dual of the homogeneous Strichartz estimate (12) we have that

$$\begin{aligned} \|\int_0^\infty S(t)|u|^2 u(t)\, dt\|_{H^1} &\lesssim \||u|^2\nabla u\|_{L^{10/7}_t L^{10/7}_x} \\ &\lesssim C\|u\|^2_{L^5_t L^5_x}\|u\|_{L^{10/3}_t W^{1,10/3}_x} \lesssim \|u\|^3_{S^1}. \end{aligned}$$

Clearly to conclude it would be enough to show that $\|u\|_{S^1} \leq C$. This is in fact proved in the following proposition.

$\square$

Proposition 6.16. *Assume that u is the H^1 global solution to the cubic, defocusing NLS in $\mathbb{R}^3$. Then*

$$\|u\|_{S^1} \leq C.$$

Proof. We first observe that (116) provides a bound in $L^4_t L^4_x$. It is to be noted that in $\mathbb{R}^3$ this norm is not an admissible Strichartz norm so we need to do a bit more work. We start by picking $\epsilon \ll 1$ to be defined later and intervals of time I_k, $k = 1, ..., M < \infty$ such that

$$\|u\|_{L^4_{I_k} L^4_x} \leq \epsilon, \tag{126}$$

for all $k = 1, ..., M$. We now work on each separate interval and at the end we put everything back together. Since for now I_k is fixed we drop the index k and we set $I = [a, b]$. By the Duhamel principle and (12) and (13) we have as above

$$\|u\|_{S^1_I} \lesssim \|u(a)\|_{H^1} + \|u\|^2_{L^5_I L^5_x} \|u\|_{L_I^{10/3} W_x^{1,10/3}}. \tag{127}$$

It is important to notice that $10/3 < 4 < 5 < 10$, where $(10/3, 10/3)$ is an admissible pair in the L^2 sense and $(10, 10)$ is admissible in the H^1 sense since by Sobolev

$$\|u\|_{L^{10}_t L^{10}_x} \leq \|u\|_{L^{10}_t W_x^{1,30/13}},$$

and $(10, 30/13)$ is an admissible pair. It follows by interpolation and (126) that

$$\|u\|_{L^5_I L^5_x} \lesssim \epsilon^\alpha \|u\|^{1-\alpha}_{S^1_I},$$

for some $\alpha > 0$. As a consequence (127) gives

$$\|u\|_{S^1_I} \lesssim \|u(a)\|_{H^1} + \epsilon^{2\alpha} \|u\|^{3-2\alpha}_{S^1_I},$$

and since the H^1 norm is uniformly bounded by energy and mass we have

$$\|u\|_{S^1_I} \lesssim 1 + \epsilon^{2\alpha} \|u\|^{3-2\alpha}_{S^1_I}. \tag{128}$$

We now use a continuity argument. Set $X(t) = \|u\|_{S^1_{[a,a+t]}}$. One can easily prove that $X(t)$ is continuous. From (128) we have

$$X(t) \lesssim 1 + \epsilon^{2\alpha} X(t)^{3-2\alpha}.$$

Then if ϵ is small enough there exist $X_0 < X_1, X_1 \gg 1$ such that either $X(t) \leq X_0$ or $X(t) \geq X_1$. But since $X(0) \lesssim 1$ and $X(t)$ is continuous it follows that $X(t) \leq X_0$ for all $t \in I$. This conclusion can be made for all I_k, $k = 1, ..., M$ and this concludes the proof.

$\square$

7. Lecture #6: Global well-posedness for the $H^1(\mathbb{R}^n)$ critical NLS -Part I

We recall that the H^1 critical exponent for (1) is $p = 1 + \frac{4}{n-2}$. We also recall the following theorem that can be basically completely proved using either directly or indirectly theorems and arguments already presented in Lecture #4 and Lecture #5:

Theorem 7.1 (Local or global small data well-posedness for the H^1 critical NLS)**.** *We have the following two results:*

1. *For any $u_0 \in H^1$ there exist $T = T(u_0)$ and a unique solution $u \in S^1_{[T,T]}$ to (1) with $p = 1 + \frac{4}{n-2}$ and $\mu = \pm 1$. Moreover there is continuity with respect to the initial data.*
2. *There exists ϵ small enough such that for any u_0, $\|u_0\|_{H^1} \leq \epsilon$ there exists a unique global solution $u \in S^1$ to (1) with $p = 1 + \frac{4}{n-2}$ and $\mu = \pm 1$. Moreover there is continuity with respect to the initial data and scattering in the sense that there exists $u_\pm \in H^1$ such that*
$$\|u(t) - S(t)u_\pm\|_{H^1} \longrightarrow 0 \quad as \quad t \to \pm\infty.$$

Proof. It is clear that the part about well-posedness is a summary of what has been proved in Lecture #4. The part about scattering instead can be proved as in Lecture #5 and by simply observing that Proposition 6.16 follows directly from the well-posedness proof thanks to the small data assumption. □

Remark 7.2. We first remark that this theorem doesn't *see* the focusing or defocusing nature of the equation. This clearly means that in Theorem 7.1 the NLS is treated as a "small" perturbation of the linear problem. Due to the criticality of the problem and hence the fact that T depends also on the profile of the initial data an iteration argument based on the conservation of mass and energy is not possible. It is also clear that even increasing the regularity of the data the large data problem doesn't become any easier.

The first break through on this problem is due to Bourgain [13]. He considers the defocusing case with $n = 3, 4$ and assumes radial symmetry for the problem. He proves the second part of Theorem 7.1 for arbitrarily large radially symmetric data. Here we summarize the main steps of Bourgain's proof for $n = 3$, which doesn't really do justice to the novelty and depth of the proof itself. The background argument is done by induction on the size of the energy E, the only quantity, besides the mass that here doesn't play much of a role, that remains controlled over time. From Theorem 7.1 the first step of the induction (small E) is in place. Let's now assume the second induction assumption that if $E < E_0$, for E_0 arbitrarily large, then the theorem is true. We take $E = E_0$ and we want to prove that also in this case the theorem is true. One first shows that the theorem follows if and only if the norm $L^{10}_t L^{10}_x$ of the solution remains bounded (see Theorem 4.8). Then the proof proceeds by contradiction. One supposes that there is a solution u such that $\|u\|_{L^{10}_t L^{10}_x}$ is arbitrarily large and $E = E_0$. The heart of the proof is on showing that at some time t_0 there is concentration of the H^1 norm: there exists a small ball B_0 centered at the origin such that $\|u(t_0)\|_{H^1(B_0)} > \delta$, and this ball is "sufficiently isolated" from the rest of the solution . It is here that the radial assumption is used. At this point one restarts the evolution at time t_0 by splitting the data as

$$\psi_0 = u(t_0)\chi_{B_0} \quad \text{and} \quad \psi_1 = u(t_0)(1 - \chi_{B_0}),$$

where χ_{B_0} is the indicator function for the ball B_0, and evolving ψ_0 with NLS and ψ_1 with a difference equation so that the sum of the two evolutions give the solution to NLS. Since now $\psi_0 \in H^1$ and $x\psi \in L^2$ it follows[25] that the evolution v of ψ_0 is global in time. Moreover

[25] This result is for example proved in [19] as a consequence of the pseudo-conformal transformation and a monotonicity formula linked to it.

since $E(\psi_0) \sim \delta^2$ it follows that $E(\psi_1) < E_0 - \delta^2$. Hence for the difference equation we are in the induction assumption. This is not quite like to have the equation under the induction assumption, but with some relatively straightforward perturbation theory[26] one also gets that the evolution w of ψ_1 is global. Hence we have a global evolution for the solution $u = v + w$ to NLS and as a consequence a uniform bound for $\|u\|_{L^{10}_t L^{10}_x}$ which is a contradiction.

Almost at the same time, with the same radial symmetry assumption above, Grillakis [41] proved a slighter weaker result than Bourgain's, namely existence and uniqueness for smooth global solution. It took few more years to remove the radial assumption and obtain the following theorem and its corollary [29]:

Theorem 7.3. *For any u_0 with finite energy, $E(u_0) < \infty$, there exists a unique[27] global solution $u \in C^0_t(\dot{H}^1_x) \cap L^{10}_{t,x}$ to (1) with $p = 5, n = 3, \mu = 1$ such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10} \, dxdt \leq C(E(u_0)). \tag{129}$$

for some constant $C(E(u_0))$ that depends only on the energy.

As one can see from Theorem 4.8 and from the arguments in Lecture #5, the $L^{10}_{t,x}$ bound above also gives scattering and and persistence of regularity:

Corollary 7.4. *Let u_0 have finite energy. Then there exist finite energy solutions $u_\pm(t,x)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$ such that*

$$\|u_\pm(t) - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

Furthermore, the maps $u_0 \mapsto u_\pm(0)$ are homeomorphisms from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$. Finally, if $u_0 \in H^s$ for some $s > 1$, then $u(t) \in H^s$ for all time t, and one has the uniform bound

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C(E(u_0), s)\|u_0\|_{H^s}.$$

Most of the rest of this lecture and Lecture # 7 will be devoted to give an idea of the proof for Theorem 7.3. Still for the defocusing case and for $n > 3$ we recall first the result of Tao [72], where an equivalent of Theorem 7.3 is proved still under the radial assumption, the result of Ryckman and Visan [68] for $n = 4$, where the radial assumption is removed, and finally the full generalization for any $n \geq 5$ by Visan [75].

The situation in the focusing case was first considered successfully by Kenig and Merle. They prove the following theorem [50]:

Theorem 7.5. *Assume that $E(u_0) < E(W), \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, where $n = 3, 4, 5$ and u_0 is radial and W is the stationary solution (soliton). Then the solution u to the critical H^1 focusing IVP (1) with data u_0 at $t = 0$ is defined for all time and there exists $u_\pm \in \dot{H}^1$ such that*

$$\|S(t)u_\pm - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

Moreover for u_0 radial, $E(u_0) < E(W)$, but $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, the solution must break down in finite time.

This result has been extend in every dimension $n \geq 3$ and for general data in [55]. Moreover a similar result has been proved by Kenig and Merle for the critical wave equation without the radial assumption [51], see also [52]. The proof of Theorem 7.5 introduces a new point of view for these problems. Using a concentration-compactness argument the authors reduce matters to a

[26] That works tanks to the fact that the ball is "sufficiently" isolatd from the rest of the solution.

[27] In fact, uniqueness actually holds in the larger space $C^0_t(\dot{H}^1_x)$ (thus eliminating the constraint that $u \in L^{10}_{t,x}$) [29].

rigidity theorem, which is proved with the aid of a localized virial identity (in the spirit of Merle [60, 61]). The radiality enters only in the proof of the rigidity theorem. In the case of the critical wave equation other consideration of *elliptic nature* are used to remove the radial assumption. The authors also use in their approach a profile decomposition proved in the context of the Schrödinger equation by Keraani [53]. For a more elaborate discussion one should consult [56].

7.6. **Idea of the proof of Theorem 7.3.** To give a complete proof of this theorem in less than two lectures is impossible, so we will first outline the idea of the proof and then we only show rigourously few parts of it.

First the naive approach: we follow the strategy of induction/contraddiction introduced by Bourgain. We define E_{crit} the critical energy below which the $L^{10}_t L^{10}_x$ norm of a solutions stays bounded by some constant depending on the energy. We then identify a smooth *minimal energy blow up solution* u of energy E_{crit} such that

$$\|u\|_{L^{10}_t L^{10}_x} > M, \tag{130}$$

where M is as large as we please. For this solution we then show a series of properties that at the end will actually give

$$\|u\|_{L^{10}_t L^{10}_x} \leq C(E_{crit}), \tag{131}$$

contradicting (130).

This is in order the summary of the properties we prove for the *minimal energy blow up solution* on a fixed (compact) interval of time I:

(1) **Frequency and space localization:** For each $t \in I$ there exists $N(t) > 0$ and $x(t) \in \mathbb{R}^3$ such that $\hat{u}(t)$ is *mostly* supported at frequency of size proportional to $N(t)$ and $u(t)$ is mostly supported on a ball centered at $x(t)$ and radius proportional to $\frac{1}{N(t)}$. To prove the frequency localization part one uses the intuition that the *minimal energy blow up solution* u, at a given time t_0, cannot have two components u_- and u_+ which Fourier transforms are supported respectively in $|\xi| \leq N$ and $|\xi| \geq KN$, $K \ll 1$, and such that both pieces carrie a large amount of energy. The reason for this is that the energy relative to u_- will make the energy relative to u_+ smaller than E_{crit} and viceversa. Hence both u_- and u_+ can flow globally. On the other hand if K is large enough their nonlinear interaction is basically negligible, hence perturbation theory says that $u \sim u_- + u_+$, hence u exists globally and its $L^{10}_t L^{10}_x$ norm is uniformly bounded, a contradiction. A similar, but just a bit more complicated, argument gives also space localization.

(2) **Frequency localized interaction Morawetz inequality:** As we mentioned several times whenever a problem is not a perturbation of the linear one, like the critical ones for example, in order to obtain a global statement we need to have a global space-time bound. We learned that the Morawetz estimates for the defocusing problem and the Viriel identity for the focusing one are the types of estimates that we want to have. Bourgain in fact used the classical Morawetz estimate that appears in (30) with $p = 5$. Here the presence of the denominator forced the radial symmetry. In our argument instead we would like to use the Interaction Morawetz estimate (116). This is weaker in the sense that we only have the fourth power, but it is also stronger since we do not have a denominator. We keep in mind that our final goal is to show boundedness of the $L^{10}_I L^{10}_x$ norm of the *minimal energy blow up solution* u so we need to upgrade the $L^4_I L^4_x$ norm. We believe that for the low frequencies, where the energy is very small thanks to localization, Strichartz estimates will be enough to give us the bound in the $L^{10}_I L^{10}_x$ norm. For the high frequencies we also have small energy, but we expect that

the Strichartz estimates are too weak here. So the idea is to first prove (116) for the high frequency part of the solution. We have for all $N_* < N_{min}$

$$\int_I \int |P_{\geq N_*} u(t,x)|^4 \, dxdt \lesssim \eta_1 N_*^{-3}, \tag{132}$$

where $N_{min} = \inf_{t \in I} N(t)$ for which one can prove $N_{min} > 0$ and η_1 is a small quantity. Note that the quantities appearing in the right hand side of (132) are independent of I.

(3) **Uniform boudedness of time interval I:** Assuming that $N(t)$ doesn't run to infinity, use the $L^4_I L^4_x$ bound, which is uniform in I, to get a uniform bound on the length of time interval I itself. With this information now, since most of the solution remains on a uniformly bounded frequency window, perturbation will provide the final uniform bound for the $L^{10}_I L^{10}_x$ norm.

(4) **Uniform Boudedness of $N(t)$:** We mentioned above that there exists N_{min} such that $0 < N_{min} \leq N(t)$, and this in not hard to prove. In fact by rescaling[28] one can assume that

$$N_{min} = 1.$$

The difficult part is to show that there exists $N_{max} < \infty$ such that

$$N(t) \leq N_{max}.$$

Again by contradiction one assumes that given $R \ll 1$ there exists t_R such that $N(t_R) > R$ and by definition most of the energy is located on frequencies $R < N(t_R) \lesssim |\xi|$. But then one can prove by a simple application of the "I-method" that although the energy has migrated on very large frequencies, some *littering* of mass has been left on medium frequencies. But mass on medium frequencies is equivalent to energy, hence there is some significant energy left over on medium frequencies. If then R is large enough these two pieces of the solution u, the one at very high frequencies and the one at medium frequencies, are very separated and each has a significant amount of energy. But this cannot happen for an *energy critical blow up solution*, as discussed above. Hence N_{max} must be bounded.

In order to proceed with the outline given above we use heavily Strichartz estimates (12) and (13), the improved bilinear estimate (14) and multilinear estimates of different kinds. A very important tool that was mentioned often above is the theory of perturbation that in practice is made of a serious of perturbation lemmas. These lemmas are particularly useful when we have to claim that if u is a solution to NLS and v is a solution to an equation which is a small perturbation of NLS, then u and v are close to each other and if one exists the other does too. Here we report two examples of such lemmas.

Lemma 7.7 (Short-time perturbations)**.** *Let I be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which is a near-solution to* (1) *with $p = 5$ and $\mu = 1$ in the sense that*

$$(i\partial_t + \frac{1}{2}\Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} + e \tag{133}$$

for some function e. Suppose that we also have the energy bound

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I \times \mathbb{R}^3)} \leq E$$

for some $E > 0$. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E' \tag{134}$$

[28]Since the problem is H^1 critical and we only use the energy, nothing will change by rescaling!

for some $E' > 0$. Assume also that we have the smallness conditions

$$\|\nabla \tilde{u}\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \epsilon_0 \tag{135}$$

$$\|\nabla e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \epsilon \tag{136}$$

$$\|\nabla e\|_{L_t^2 L_x^{6/5}} \leq \epsilon \tag{137}$$

for some $0 < \epsilon < \epsilon_0$, where ϵ_0 is some constant $\epsilon_0 = \epsilon_0(E, E') > 0$.

We conclude that there exists a solution u to (1) with $p = 5$ and $\mu = 1$ on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at t_0, and furthermore

$$\|u - \tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E' \tag{138}$$

$$\|u\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E' + E \tag{139}$$

$$\|u - \tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \lesssim \|\nabla(u - \tilde{u})\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \lesssim \epsilon \tag{140}$$

$$\|\nabla(i\partial_t + \frac{1}{2}\Delta)(u - \tilde{u})\|_{L_t^2 L_x^{6/5}(I \times \mathbb{R}^3)} \lesssim \epsilon. \tag{141}$$

Note that $u(t_0) - \tilde{u}(t_0)$ is allowed to have large energy, albeit at the cost of forcing ϵ to be smaller, and worsening the bounds in (138). From the Strichartz estimate (12), we see that the hypothesis (136) is redundant if one is willing to take $E' = O(\varepsilon)$.

Proof. By the well-posedness theory presented in Lecture #4, it suffice to prove (138) - (141) as *a priori* estimates[29]. We establish these bounds for $t \geq t_0$, since the corresponding bounds for the $t \leq t_0$ portion of I are proved similarly.

First note that the Strichartz estimate (12) and (13) give,

$$\|\tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E + \|\tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \cdot \|\tilde{u}\|^4_{\dot{S}^1(I \times \mathbb{R}^3)} + \varepsilon.$$

By (135) and Sobolev embedding we have $\|\tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \lesssim \varepsilon_0$. A standard continuity argument in I then gives (if ε_0 is sufficiently small depending on E)

$$\|\tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E. \tag{142}$$

Define $v := u - \tilde{u}$. For each $t \in I$ define the quantity

$$S(t) := \|\nabla(i\partial_t + \frac{1}{2}\Delta)v\|_{L_t^2 L_x^{6/5}([t_0,t] \times \mathbb{R}^3)}.$$

From using again Strichartz estimates and the definition of S^1, (136), we have

$$\begin{aligned}
\|\nabla v\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} &\lesssim \|\nabla(v - e^{i(t-t_0)\frac{1}{2}\Delta} v(t_0))\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} \qquad (143)\\
&\quad + \|\nabla e^{i(t-t_0)\frac{1}{2}\Delta} v(t_0)\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} \\
&\lesssim \|v - e^{i(t-t_0)\frac{1}{2}\Delta} v(t_0)\|_{\dot{S}^1([t_0,t] \times \mathbb{R}^3)} + \varepsilon \\
&\lesssim S(t) + \varepsilon. \qquad (144)
\end{aligned}$$

On the other hand, since v obeys the equation

$$(i\partial_t + \frac{1}{2}\Delta)v = |\tilde{u} + v|^4(\tilde{u} + v) - |\tilde{u}|^4 \tilde{u} - e = \sum_{j=1}^{5} \emptyset(v^j \tilde{u}^{5-j}) - e$$

[29] That is, we may assume the solution u already exists and is smooth on the entire interval I.

where $\emptyset(v_1, v_2, v_3, v_4, v_5)$ denotes any combination of v_i and $\bar{v_j}$. By some standard multilinear estimates, (135), (137), (144) then

$$S(t) \lesssim \varepsilon + \sum_{j=1}^{5} (S(t) + \varepsilon)^j \varepsilon_0^{5-j}.$$

If ε_0 is sufficiently small, a standard continuity argument then yields the bound $S(t) \lesssim \varepsilon$ for all $t \in I$. This gives (141), and (140) follows from (144). Applying Strichartz inequalities again, (134) we then conclude (138) (if ε is sufficiently small), and then from (142) and the triangle inequality we conclude (139). □

We will actually be more interested in iterating the above lemma to deal with the more general situation of near-solutions with finite but arbitrarily large $L^{10}_{t,x}$ norms.

Lemma 7.8 (Long-time perturbations). *Let I be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which obeys the bounds*

$$\|\tilde{u}\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \leq M \tag{145}$$

and

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I \times \mathbb{R}^3)} \leq E \tag{146}$$

for some $M, E > 0$. Suppose also that $\tilde{u}$ is a near-solution to (1) with $p = 5$ and $\mu = 1$ in the sense that it solves (133) for some e. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E'$$

for some $E' > 0$. Assume also that we have the smallness conditions,

$$\|\nabla e^{i(t-t_0)\frac{1}{2}\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^{10}_t L^{30/13}_x(I \times \mathbb{R}^3)} \leq \varepsilon \tag{147}$$
$$\|\nabla e\|_{L^2_t L^{6/5}_x(I \times \mathbb{R}^3)} \leq \varepsilon$$

for $0 < \varepsilon < \varepsilon_1$, where ε_1 is some constant $\varepsilon_1 = \varepsilon_1(E, E', M) > 0$. We conclude there exists a solution u to (1) with $p = 5$ and $\mu = 1$ on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at t_0, and furthermore

$$\|u - \tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \leq C(M, E, E')$$
$$\|u\|_{\dot{S}^1(I \times \mathbb{R}^3)} \leq C(M, E, E')$$
$$\|u - \tilde{u}\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \leq \|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I \times \mathbb{R}^3)} \leq C(M, E, E')\varepsilon.$$

Once again, the hypothesis (147) is redundant by the Strichartz estimate if one is willing to take $E' = O(\varepsilon)$; however it will be useful in our applications to know that this Lemma can tolerate a perturbation which is large in the energy norm but whose free evolution is small in the $L^{10}_t \dot{W}^{1,30/13}_x$ norm.

This lemma is already useful in the $e = 0$ case, as it says that one has local well-posedness in the energy space whenever the $L^{10}_{t,x}$ norm is bounded; in fact one has locally Lipschitz dependence on the initial data. For similar perturbative results see [13], [12].

Proof. As in the previous proof, we may assume that t_0 is the lower bound of the interval I. Let $\varepsilon_0 = \varepsilon_0(E, 2E')$ be as in Lemma 7.7. (We need to replace E' by the slightly larger $2E'$ as the $\dot{H}^1$ norm of $u - \tilde{u}$ is going to grow slightly in time.)

The first step is to establish a $\dot{S}^1$ bound on $\tilde{u}$. Using (145) we may subdivide I into $C(M, \varepsilon_0)$ time intervals such that the $L^{10}_{t,x}$ norm of $\tilde{u}$ is at most ε_0 on each such interval. By using (146) and Strichartz estimates, as in the proof of (142), we see that the $\dot{S}^1$ norm of $\tilde{u}$ is $O(E)$ on each of these intervals. Summing up over all the intervals we conclude

$$\|\tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \le C(M,E,\varepsilon_0)$$

and in particular

$$\|\nabla\tilde{u}\|_{L_t^{10}L_x^{30/13}(I\times\mathbb{R}^3)} \le C(M,E,\varepsilon_0).$$

We can then subdivide the interval I into $N \le C(M,E,\varepsilon_0)$ subintervals $I_j \equiv [T_j, T_{j+1}]$ so that on each I_j we have,

$$\|\nabla\tilde{u}\|_{L_t^{10}L_x^{30/13}(I_j\times\mathbb{R}^3)} \le \varepsilon_0.$$

We can then verify inductively using Lemma 7.7 for each j that if ε_1 is sufficiently small depending on ε_0, N, E, E', then we have

$$\begin{aligned}
\|u-\tilde{u}\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\le C(j)E'\\
\|u\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\le C(j)(E'+E)\\
\|\nabla(u-\tilde{u})\|_{L_t^{10}L_x^{30/13}(I_j\times\mathbb{R}^3)} &\le C(j)\varepsilon\\
\|\nabla(i\partial_t+\frac{1}{2}\Delta)(u-\tilde{u})\|_{L_t^{2}L_x^{6/5}(I_j\times\mathbb{R}^3)} &\le C(j)\varepsilon
\end{aligned}$$

and hence by Strichartz we have

$$\begin{aligned}
&\|\nabla e^{i(t-T_{j+1})\frac{1}{2}\Delta}(u(T_{j+1})-\tilde{u}(T_{j+1}))\|_{L_t^{10}L_x^{30/13}(I\times\mathbb{R}^3)}\\
&\le \|\nabla e^{i(t-T_j)\frac{1}{2}\Delta}(u(T_j)-\tilde{u}(T_j))\|_{L_t^{10}L_x^{30/13}(I\times\mathbb{R}^3)} + C(j)\varepsilon
\end{aligned}$$

and

$$\|u(T_{j+1})-\tilde{u}(T_{j+1})\|_{\dot{H}^1} \le \|u(T_j)-\tilde{u}(T_j)\|_{\dot{H}^1} + C(j)\varepsilon$$

allowing one to continue the induction (if ε_1 is sufficiently small depending on E, N, E', ε_0, then the quantity in (134) will not exceed $2E'$). The claim follows. □

8. Lecture #7:Global well-posedness for the $H^1(\mathbb{R}^n)$ critical NLS -Part II

We start by recalling the critical H^1 defocusing IVP for which we want to prove global well-posedness and scattering for large data:

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = |u|^4 u \\ u(0,x) = u_0(x). \end{cases} \tag{148}$$

where $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R}_t \times \mathbb{R}^3_x$. This equation has as Hamiltonian,

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx. \tag{149}$$

We now outline the proof of Theorem 7.3, breaking it down into a number of smaller propositions.

8.1. **Zeroth stage: Induction on energy.** The first observation is that in order to prove Theorem 7.3, it suffices to do so for Schwartz solutions. Indeed, once one obtains a uniform $L^{10}_{t,x}(I \times \mathbb{R}^3)$ bound for all Schwartz solutions and all compact I, one can then approximate arbitrary finite energy initial data by Schwartz initial data and use Lemma 7.8 to show that the corresponding sequence of solutions to (148) converge in the homogeneous Strichartz space $\dot{S}^1(I \times \mathbb{R}^3)$ to a finite energy solution to (148). We omit the standard details.

For every energy $E \geq 0$ we define the quantity $0 \leq M(E) \leq +\infty$ by

$$M(E) := \sup\{\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)}\}$$

where $I_* \subset \mathbb{R}$ ranges over all compact time intervals, and u ranges over all Schwartz solutions to (148) on $I_* \times \mathbb{R}^3$ with $E(u) \leq E$. We shall adopt the convention that $M(E) = 0$ for $E < 0$. By the above discussion, it suffices to show that $M(E)$ is finite for all E.

In the argument of Bourgain [13] (see also [12]), the finiteness of $M(E)$ in the spherically symmetric case is obtained by an induction on the energy E; indeed a bound of the form

$$M(E) \leq C(E, \eta, M(E - \eta^4))$$

is obtained for some explicit $0 < \eta = \eta(E) \ll 1$ which does not collapse to 0 for any finite E, and this easily implies via induction that $M(E)$ is finite for all E. Our argument will follow a similar induction on energy strategy, however it will be convenient to run this induction in the contrapositive, assuming for contradiction that $M(E)$ can be infinite, studying the minimal energy E_{crit} for which this is true, and then obtaining a contradiction using the "induction hypothesis" that $M(E)$ is finite for all $E < E_{crit}$. This will be more convenient for us, especially as we will require more than one small parameter η.

We turn to the details. We assume for contradiction that $M(E)$ is not always finite. From Lemma 7.8 we see that the set $\{E : M(E) < \infty\}$ is open; clearly it is also connected and contains 0. By our contradiction hypothesis, there must therefore exist a *critical energy* $0 < E_{crit} < \infty$ such that $M(E_{crit}) = +\infty$, but $M(E) < \infty$ for all $E < E_{crit}$. One can think of E_{crit} as the minimal energy required to create a blowup solution. For instance, we have

Lemma 8.2 (Induction on energy hypothesis)**.** *Let $t_0 \in \mathbb{R}$, and let $v(t_0)$ be a Schwartz function such that $E(v(t_0)) \leq E_{crit} - \eta$ for some $\eta > 0$. Then there exists a Schwartz global solution $v : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}$ to (148) with initial data $v(t_0)$ at time $t = t_0$ such that $\|v\|_{L^{10}_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \leq M(E_{crit} - \eta) = C(\eta)$. Furthermore we have $\|v\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^3)} \leq C(\eta)$.*

Indeed, this Lemma follows immediately from the definition of E_{crit} and Theorem 4.8. As in the argument in [13], we will need a small parameter $0 < \eta = \eta(E_{crit}) \ll 1$ depending on E_{crit}.

In fact, our argument is somewhat lengthy and we actually need to use *seven* such parameters

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 \gg \eta_5 \gg \eta_6 > 0.$$

In the proof we always assume implicitly that each η_j has been chosen to be sufficiently small depending on the previous parameters. We also often display the dependence of constants on a parameter, e.g. $C(\eta)$ denotes a large constant depending on η, and $c(\eta)$ denotes a small constant depending upon η. When $\eta_1 \gg \eta_2$, we understand $c(\eta_1) \gg c(\eta_2)$ and $C(\eta_1) \ll C(\eta_2)$.

Since $M(E_{crit})$ is infinite, it is in particular larger than $1/\eta_6$. By definition of M, this means that we may find a compact interval $I_* \subset \mathbb{R}$ and a smooth solution $u : I_* \times \mathbb{R}^3 \to \mathbb{C}$ to (148) with $E_{crit}/2 \le E(u) \le E_{crit}$ so that u is ridiculously large in the sense that

$$\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} > 1/\eta_6. \tag{150}$$

The main argument then consists in showing that this leads to a contradiction[30]. Although u does not actually blow up (it is assumed smooth on all of the compact interval I_*), it is still convenient to think of u as almost[31] blowing up in $L^{10}_{t,x}$ in the sense of (150). We summarize the above discussion with the following,

Definition 8.3. A *minimal energy blowup solution* of (148) is a Schwartz solution on a time interval I_* with energy[32],

$$\frac{1}{2}E_{crit} \le E(u)(t) = \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx \le E_{crit} \tag{151}$$

and $L^{10}_{x,t}$ norm enormous in the sense of (150).

We remark that both conditions (150), (151) are invariant under the scaling (66) (though of course the interval I_* will be dilated by μ^2 under this scaling). Thus applying the scaling (66) to a minimal energy blowup solution produces another minimal energy blowup solution. Some of the proofs of the sub-propositions below will revolve around a specific frequency N; using this scale invariance, we can then normalize that frequency to equal 1 for the duration of that proof. (Different parts of the argument involve different key frequencies, but we will not run into problems because we will only normalize one frequency at a time).

Henceforth we will not mention the E_{crit} dependence of our constants explicitly, as all our constants will depend on E_{crit}. We shall need however to keep careful track of the dependence of our argument on $\eta_0, \dots, \eta_6$. Broadly speaking, we will start with the largest η, namely η_0, and slowly "retreat" to increasingly smaller values of η as the argument progresses (such a retreat will for instance usually be required whenever the induction hypothesis Lemma 8.2 is invoked). However we will only retreat as far as η_5, not η_6, so that (150) will eventually lead to a contradiction when we show that

$$\|u\|_{L^{10}_{t,x}(I^* \times \mathbb{R}^3)} \le C(\eta_0, \dots, \eta_5).$$

---

[30] Assuming, of course, that the parameters $\eta_0, \dots, \eta_6$ are each chosen to be sufficiently small depending on previous parameters. It is important to note however that the η_j cannot be chosen to be small depending on the interval I_* or the solution u; our estimates must be uniform with respect to these parameters.

[31] For instance, u might genuinely blow up at some time $T_* > 0$, but I_* is of the form $I_* = [0, T_* - \varepsilon]$ for some very small $0 < \varepsilon \ll 1$, and thus u remains Schwartz on $I_* \times \mathbb{R}^3$.

[32] We could modify our arguments below to allow the assumption here $E(u) = E_{crit}$. For example, the arguments in the proof of Proposition 8.5 below also show that the function $\tilde{M}(s) := \sup_{E(u)=s}\{\|u\|_{L^{10}_{x,t}}\}$ is a nondecreasing function of s. On first reading, the reader may imagine $E(u) = E_{crit}$ in Definition 8.3.

Together with our assumption that we are considering a minimal energy blowup solution u as in Definition 8.3, Sobolev embedding implies the bounds on kinetic energy

$$\|u\|_{L_t^\infty \dot{H}_x^1(I_* \times \mathbb{R}^3)} \sim 1 \tag{152}$$

and potential energy

$$\|u\|_{L_t^\infty L_x^6(I_* \times \mathbb{R}^3)} \lesssim 1 \tag{153}$$

(since our implicit constants are allowed to depend on E_{crit}). Note that we do not presently have any *lower* bounds on the potential energy, but see below.

Having displayed our preliminary bounds on the kinetic and potential energy, we briefly discuss the mass $\int_{\mathbb{R}^3} |u(t,x)|^2\, dx$, which is another conserved quantity. Because of our *a priori* assumption that u is Schwartz, we know that this mass is finite. However, we cannot obtain uniform control on this mass using our bounded energy assumption, because the very low frequencies of u may simultaneously have very small energy and very large mass. Furthermore it is dangerous to rely too much on this conserved mass for this energy-critical problem as the mass is not invariant under the natural scaling (66) of the equation (indeed, it is super-critical with respect to that scaling). On the other hand, from (152) we know that the *high frequencies* of u have small mass:

$$\|P_{>M} u\|_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{M} \text{ for all } M \in 2^{\mathbb{Z}}. \tag{154}$$

Thus we will still be able to use the concept of mass in our estimates as long as we restrict our attention to sufficiently high frequencies.

8.4. **First stage: Localization control on u.** We aim to show that a*minimal energy blowup solution* as in Definition 8.3 does not exist. Intuitively, as we already mentioned in Lecture #6, it seems reasonable to expect that a *minimal-energy blowup solution* should be "irreducible" in the sense that it cannot be decoupled into two or more components of strictly smaller energy that essentially do not interact with each other (i.e. each component also evolves via (148) modulo small errors), since one of the components must then also blow up, contradicting the minimal-energy hypothesis. In particular, we expect at every time that such a solution should be localized in both frequency and space.

The first main step in the proof of Theorem 7.3 is to make the above heuristics rigorous for our solution u. Roughly speaking, we would like to assert that at each time t, the solution $u(t)$ is localized in both space and frequency to the maximum extent allowable under the uncertainty principle (i.e. if the frequency is localized to $N(t)$, we would like to localize $u(t)$ spatially to the scale $1/N(t)$).

These sorts of localizations already appear for instance in the argument of Bourgain [13], [12], where the induction on energy argument is introduced. Informally[33], the reason that we can expect such localization is as follows. Suppose for contradiction that at some time t_0 the solution $u(t_0)$ can be split into two parts $u(t_0) = v(t_0) + w(t_0)$ which are widely separated in either space or frequency, and which each carry a nontrivial amount $O(\eta^C)$ of energy for some $\eta_5 \le \eta \le \eta_0$. Then by orthogonality we expect v and w to each have strictly smaller energy than u, e.g. $E(v(t_0)), E(w(t_0)) \le E_{crit} - O(\eta^C)$. Thus by Lemma 8.2 we can extend $v(t)$ and $w(t)$ to

[33]The heuristic that *minimal energy blowup solutions* should be strongly localized in both space and frequency has been employed in previous literature for a wide variety of nonlinear equations, including many of elliptic or parabolic type. Our formalizations of this heuristic, however, rely on the induction on energy methods of Bourgain and perturbation theory, as opposed to variational or compactness arguments. This last one indeed is the method used in [50, 51].

all of $I_* \times \mathbb{R}^3$ by evolving the nonlinear Schrödinger equation (148) for v and w separately, and furthermore we have the bounds

$$\|v\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)}, \|w\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \le M(E_{crit} - O(\eta^C)) \le C(\eta).$$

Since v and w both solve (148) separately, and v and w were assumed to be widely separated, we thus expect $v + w$ to solve (148) *approximately.* The idea is then to use the perturbation theory from Lecture #6 to obtain a bound of the form $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \le C(\eta)$, which contradicts (150) if η_6 is sufficiently small.

As recalled in Lecture #6, model example of this type of strategy occurs in Bourgain's argument [12], where substantial effort is invested in locating a "bubble" - a small localized pocket of energy - which is sufficiently isolated in physical space from the rest of the solution. One then removes this bubble, evolves the remainder of the solution, and then uses perturbation theory, augmented with the additional information about the isolation of the bubble, to place the bubble back in. We will use arguments similar to these in the sequel, but first we need instead to show that a solution of (148) which is sufficiently delocalized in frequency space is globally spacetime bounded. More precisely, we have:

Proposition 8.5 (Frequency delocalization implies spacetime bound)**.** *Let $\eta > 0$, and suppose there exists a dyadic frequency $N_{lo} > 0$ and a time $t_0 \in I_*$ such that we have the energy separation conditions*

$$\|P_{\le N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \ge \eta \tag{155}$$

and

$$\|P_{\ge K(\eta) N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \ge \eta. \tag{156}$$

If $K(\eta)$ is sufficiently large depending on η, i.e.

$$K(\eta) \ge C(\eta)$$

then we have

$$\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \le C(\eta). \tag{157}$$

Clearly the conclusion of Proposition 8.5 is in conflict with the hypothesis (150), and so we should now expect the solution to be localized in frequency for every time t. This is indeed the case:

Corollary 8.6 (Frequency localization of energy at each time)**.** *A minimal energy blowup solution of* (148) *(see Definition 8.3) satisfies: For every time $t \in I_*$ there exists a dyadic frequency $N(t) \in 2^{\mathbb{Z}}$ such that for every $\eta_5 \le \eta \le \eta_0$ we have small energy at frequencies $\ll N(t)$,*

$$\|P_{\le c(\eta) N(t)} u(t)\|_{\dot{H}^1} \le \eta, \tag{158}$$

small energy at frequencies $\gg N(t)$,

$$\|P_{\ge C(\eta) N(t)} u(t)\|_{\dot{H}^1} \le \eta, \tag{159}$$

and large energy at frequencies $\sim N(t)$,

$$\|P_{c(\eta) N(t) < \cdot < C(\eta) N(t)} u(t)\|_{\dot{H}^1} \sim 1. \tag{160}$$

Here $0 < c(\eta) \ll 1 \ll C(\eta) < \infty$ are quantities depending on η.

Informally, this Corollary asserts that at every given time t the solution u is essentially concentrated at a single frequency $N(t)$. Note however that we do not presently have any information as to how $N(t)$ evolves in time; obtaining long-term control on $N(t)$ will be a key objective of later stages of the proof.

Proof. For each time $t \in I_*$, we define $N(t)$ as

$$N(t) := \sup\{N \in 2^{\mathbb{Z}} : \|P_{\leq N} u(t)\|_{\dot H^1} \leq \eta_0\}.$$

Since $u(t)$ is Schwartz, we see that $N(t)$ is strictly larger than zero; from the lower bound in (152) we see that $N(t)$ is finite. By definition of $N(t)$, we have

$$\|P_{\leq 2N(t)} u(t)\|_{\dot H^1} > \eta_0.$$

Now let $\eta_5 \leq \eta \leq \eta_0$. Observe that we now have (159) if $C(\eta)$ is chosen sufficiently large, because if (159) failed then Proposition 8.5 would imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (150) if η_6 is sufficiently small. In particular we have (159) for $\eta = \eta_0$. Since we also have (158) for $\eta = \eta_0$ by construction of $N(t)$, we thus see from (152) that we have (160) for $\eta = \eta_0$, which of course then implies (again by (152)) the same bound for all $\eta_5 \leq \eta \leq \eta_0$. Finally, we obtain (158) for all $\eta_5 \leq \eta \leq \eta_0$ if $c(\eta)$ is chosen sufficiently small, since if (158) failed then by combining it with (160) and Proposition 8.5 we would once again imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (150). □

Having shown that any *minimal energy blowup solution* u must be localized in frequency at each time, we now turn to showing that such a u is also localized in physical space. This turns out to be somewhat more involved, although it still follows the same general strategy. We first borrow a useful trick from [13]; since u is Schwartz, we may divide the interval I_* into three consecutive pieces $I_* := I_- \cup I_0 \cup I_+$ where each of the three intervals contains a third of the $L^{10}_{t,x}$ density:

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dxdt = \frac{1}{3} \int_{I_*} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dxdt \text{ for } I = I_-, I_0, I_+.$$

In particular from (150) we have

$$\|u\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \gtrsim 1/\eta_6 \text{ for } I = I_-, I_0, I_+. \tag{161}$$

Thus to contradict (150) it suffices to obtain $L^{10}_{t,x}$ bounds on just one of the three intervals I_-, I_0, I_+.

It is in the middle interval I_0 that we can obtain physical space localization; this shall be done in several stages. The first step is to ensure that the potential energy $\int_{\mathbb{R}^3} |u(t,x)|^6\, dx$ is bounded from below.

Proposition 8.7 (Potential energy bounded from below). *For any minimal energy blowup solution of* (148) *(see Definition 8.3) we have for all* $t \in I_0$,

$$\|u(t)\|_{L^6_x} \geq \eta_1. \tag{162}$$

The proof of this proposition is inspired by a similar argument of Bourgain [13]. Using (162) and some simple Fourier analysis, we can thus establish the following concentration result:

Proposition 8.8 (Physical space concentration of energy at each time). *Any minimal energy blowup solution of* (148) *satisfies: For every* $t \in I_0$, *there exists an* $x(t) \in \mathbb{R}^3$ *such that*

$$\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |\nabla u(t,x)|^2\, dx \gtrsim c(\eta_1) \tag{163}$$

and

$$\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |u(t,x)|^p\, dx \gtrsim c(\eta_1) N(t)^{\frac{p}{2}-3} \tag{164}$$

for all $1 < p < \infty$*, where the implicit constant can depend on* p*. In particular we have*

$$\int_{|x-x(t)|\leq C(\eta_1)/N(t)} |u(t,x)|^6 \, dx \gtrsim c(\eta_1), \tag{165}$$

Similar results were obtained in [13], [41] in the radial case; see also [9]. Informally, the above estimates assert that $u(t,x)$ is roughly of size $N(t)^{1/2}$ on the average when $|x - x(t)| \lesssim 1/N(t)$; observe that this is consistent with bounded energy (152) as well as with Corollary 8.6 and the uncertainty principle.

It turns out that in our argument, it is not enough to know that the energy concentrates at one location $x(t)$ at each time; we must also show that the energy is small at all other locations, where $|x - x(t)| \gg 1/N(t)$. The main tool for achieving this is

Proposition 8.9 (Physical space localization of energy at each time)**.** *For any minimal energy blowup solution of* (148) *we have for every* $t \in I_0$

$$\int_{|x-x(t)|>1/(\eta_2 N(t))} |\nabla u(t,x)|^2 \, dx \lesssim \eta_1. \tag{166}$$

The proof follows a similar strategy to that used to prove Corollary 8.5; the main difference is that we now consider spatially separated components of u rather than frequency separated components, and instead of using multilinear Strichartz estimates to establish the decoupling of these components, we shall rely instead on approximate finite speed of propagation and on the pseudoconformal identity.

To summarize, at each time t we have a location $x(t)$, around which the kinetic and potential energy are large, and away from which the kinetic energy is small (and one can also show the potential energy is small, although we will not need this). From this and a little Fourier analysis we obtain an important conclusion:

Proposition 8.10 (Reverse Sobolev inequality)**.** *Assuming* u *is a minimal energy blowup solution (and hence* (151)*,* (158)*-*(166) *hold), we have that for every* $t_0 \in I_0$*, any* $x_0 \in \mathbb{R}^3$*, and any* $R \geq 0$*,*

$$\int_{B(x_0,R)} |\nabla u(t_0,x)|^2 \, dx \lesssim \eta_1 + C(\eta_1,\eta_2) \int_{B(x_0,C(\eta_1,\eta_2)R)} |u(t_0,x)|^6 \, dx \tag{167}$$

Thus, up to an error of η_1, we are able to control the kinetic energy locally by the potential energy[34]. This fact will be crucial in the interaction Morawetz portion of our argument when we have an error term involving the kinetic energy, and control of a positive term which involves the potential energy; the reverse Sobolev inequality is then used to control the former by the latter.

To summarize, the statements above tell us that any minimal energy blowup solution (Definition 8.3) to the equation (148) must be localized in both frequency and physical space at every time. We are still far from done: we have not yet precluded blowup in finite time (which would happen if $N(t) \to \infty$ as $t \to T_*$ for some finite time T_*), nor have we eliminated soliton or soliton-like solutions (which would correspond, roughly speaking, to $N(t)$ staying close to constant for all time t). To achieve this we need spacetime integrability bounds on u. Our main tool for this is a frequency-localized version of the interaction Morawetz estimate (116), to which we now turn.

[34]Note that this is a special property of the *minimal energy blowup solution*, reflecting the very strong physical space localization properties of such a solution; it is false in general, even for solutions to the free Schrödinger equation. Of course, Proposition 8.7 is similarly false in general, for instance for solutions of the free Schrödinger equation, the L^6_x norm goes to zero as $t \to \pm\infty$.

8.11. **Second stage: Localized Morawetz estimate.** In order to localize the interaction Morawetz inequality, it turns out to be convenient to work at the "minimum" frequency attained by u.

We observe that

$$\|P_{c(\eta_0)N(t)<\cdot<C(\eta_0)N(t)}u(t)\|_{\dot H^1} \leq C(\eta_0)N(t)\|u\|_{L^\infty_t L^2_x}$$

Comparing this with (160) we obtain the lower bound

$$N(t) \geq c(\eta_0)\|u\|^{-1}_{L^\infty_t L^2_x}$$

for $t \in I_0$. Since u is Schwartz, the right-hand side is nonzero, and thus the quantity

$$N_{min} := \inf_{t\in I_0} N(t)$$

is strictly positive.

From (158) we see that the low frequency portion of the solution - where $|\xi| \leq c(\eta_0)N_{min}$ - has small energy; one might then hope to use Strichartz estimates to obtain some spacetime control on these low frequencies. However, we do not yet have much control on the high frequencies $|\xi| \geq c(\eta_0)N_{min}$, apart from the energy bounds (152) and (153) of course.

Our initial spacetime bound in the high frequencies is provided by the following interaction Morawetz estimate.

Proposition 8.12 (Frequency-localized interaction Morawetz estimate)**.** *Assuming u is a minimal energy blowup solution of* (148) *(and hence* (151), (158)-(167) *all hold), we have for all* $N_* < c(\eta_3)N_{min}$

$$\int_{I_0}\int |P_{\geq N_*}u(t,x)|^4 \; dxdt \lesssim \eta_1 N_*^{-3}. \tag{168}$$

Remark 8.13. The factor N_*^{-3} on the right-hand side of (168) is mandated by scale-invariance considerations (cf. (66)). The η_1 factor on the right side reflects our smallness assumption on N_*: if we think of N_* as being very small and then scale the solution so that $N_* = 1$, we are pushing the energy to very high frequencies so heuristically it's not unreasonable to expect the supercritical $L^4_{x,t}$ norm on the left hand side to be small.

Regarding the size of N_*: write for the moment $\tilde c(\eta_3)$ as the constant appearing in Corollary 8.6 with $\eta = \eta_3$. The constant $c(\eta_3)$ appearing in Proposition 8.12 is chosen so $c(\eta_3) \lesssim \tilde c(\eta_3)\cdot\eta_3$, hence at all times we know there is very little energy at frequencies below $\frac{N_*}{\eta_3}$, and (ignoring factors of N_* which can be scaled to 1) above frequency N_* there is very little (at most η_3/N_*) L^2 mass.

This small η_1 factor will be used to close a bootstrap argument in the proof of the important estimate on the movement of energy to very low frequencies.

The key thing about this estimate is that the right-hand side does not depend on I_0; thus for instance it is already useful in eliminating soliton or pseudosoliton solutions, at least for frequencies close to N_{min}. (Frequencies much larger than N_{min} still cause difficulty, and will be dealt with later in the argument). Proposition 8.12 roughly corresponds to the localized Morawetz inequality used by Bourgain [13], [12] and Grillakis [41]. The main advantage of (168) is that it is not localized to near the spatial origin, in contrast with the standard (30) Morawetz inequalities.

Although this proposition is based on the interaction Morawetz inequality developed in the references given above, there are significant technical difficulties in truncating that inequality to the high frequencies. As a consequence the proof of this proposition is somewhat involved. Also, we caution the reader that the above proposition is not proved as an *a priori* estimate;

indeed the proof relies crucially on the assumption that u is a *minimal energy blowup solution* in the sense of 8.3, and in particular verifies the reverse Sobolev inequality (167).

Combining Proposition 8.12 with Proposition 8.8 gives us the following integral bound on $N(t)$.

Corollary 8.14. *For any minimal energy blowup solution of* (148)*, we have*

$$\int_{I_0} N(t)^{-1}\ dt \lesssim C(\eta_1, \eta_3) N_{min}^{-3}. \tag{169}$$

Proof. Let $N_* := c(\eta_3) N_{min}$ for some sufficiently small $c(\eta_3)$. Then from Proposition 8.12 we have

$$\int_{I_0} \int_{\mathbb{R}^3} |P_{\geq N_*} u(t,x)|^4\ dxdt \lesssim \eta_1 N_*^{-3} \lesssim C(\eta_1, \eta_3) N_{min}^{-3}.$$

On the other hand, from Bernstein inequality and (152) we have for each $t \in I_0$ that

$$\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |P_{<N_*} u(t,x)|^4\ dx \lesssim N(t)^{-3} \|P_{<N_*} u(t)\|_{L_x^\infty}^4 \lesssim C(\eta_1) N(t)^{-3} N_*^2,$$

so by (164) and the triangle inequality we have (noting that $N_* \leq c(\eta_3) N(t)$)

$$\int_{\mathbb{R}^3} |P_{\geq N_*} u(t,x)|^4\ dx \gtrsim c(\eta_1) N(t)^{-1}.$$

Comparing this with the previous estimate, the claim follows. □

Remark 8.15. The estimate (169) is scale-invariant under the natural scaling (32) (N has the units of $length^{-1}$, and t has the units of $length^2$). In the radial case, a somewhat similar estimate was obtained by Bourgain [13] and implicitly also by Grillakis [41]; in our notation, this bound would be the assertion that

$$\int_I N(t)\ dt \lesssim |I|^{1/2} \tag{170}$$

for all $I \subseteq I_0$; indeed in the radial case (when $x(t) = 0$) this bound easily follows from Proposition 8.8 and a local version of (30). Both estimates are equally good at estimating the amount of time for which $N(t)$ is comparable to N_{min}, but Corollary 8.14 is much weaker than (170) when it comes to controlling the times for which $N(t) \gg N_{min}$. Indeed if we could extend (170) to the nonradial case one could obtain a significantly shorter proof of Theorem 7.3, however we were unable to prove this bound directly, although it can be deduced from Corollary 8.14 and Proposition 8.19 below).

This Corollary allows us to obtain some useful $L^{10}_{t,x}$ bounds in the case when $N(t)$ is bounded from above.

Corollary 8.16 (Nonconcentration implies spacetime bound)**.** *Let $I \subseteq I_0$, and suppose there exists a $N_{max} > 0$ such that $N(t) \leq N_{max}$ for all $t \in I$. Then for any localized minimal energy blowup solution of* (148) *we have*

$$\|u\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{max}/N_{min})$$

and furthermore

$$\|u\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{max}/N_{min}).$$

Proof. We may use scale invariance (32) to rescale $N_{min} = 1$. From Corollary 8.14 we obtain the useful bound

$$|I| \lesssim C(\eta_1, \eta_3, N_{max}).$$

Let $\delta = \delta(\eta_0, N_{max}) > 0$ be a small number to be chosen later. We may partition I into $O(|I|/\delta)$ intervals $I_1, \ldots, I_J$ of length at most δ. Let I_j be any of these intervals, and let t_j be any time in I_j. Observe from Corollary 8.6 and the hypothesis $N(t_j) \leq N_{max}$ that

$$\|P_{\geq C(\eta_0)N_{max}} u(t_j)\|_{\dot{H}^1} \leq \eta_0$$

(for instance). Now let $\tilde{u}(t) := e^{i(t-t_j)\Delta} P_{<C(\eta_0)N_{max}} u(t_j)$ be the free evolution of the low and medium frequencies of u. The above estimate then becomes

$$\|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}^1} \leq \eta_0.$$

On the other hand, from Bernstein inequality and (152) we have

$$\|\tilde{u}(t)\|_{L^{10}_x} \lesssim C(\eta_0, N_{max}) \|\tilde{u}(t_j)\|_{\dot{H}^1} \lesssim C(\eta_0, N_{max})$$

for all $t \in I_j$, and hence

$$\|\tilde{u}\|_{L^{10}_{t,x}(I_j \times \mathbb{R}^3)} \lesssim C(\eta_0, N_{max}) \delta^{1/10}.$$

Similarly we have

$$\|\nabla(|\tilde{u}(t)|^4 \tilde{u}(t))\|_{L^{6/5}_x} \lesssim \|\nabla \tilde{u}(t)\|_{L^6_x} \|\tilde{u}(t)\|^4_{L^6_x} \lesssim C(\eta_0, N_{max}) \|\tilde{u}(t_j)\|^5_{\dot{H}^1} \lesssim C(\eta_0, N_{max})$$

and hence

$$\|\nabla(|\tilde{u}(t)|^4 \tilde{u}(t))\|_{L^2_t L^{6/5}_x(I_j \times \mathbb{R}^3)} \lesssim C(\eta_0, N_{max}) \delta^{1/2}.$$

From these two estimates, the energy bound (152), and Lemma 7.7 with $e = -|\tilde{u}|^4 \tilde{u}$, we see (if δ is chosen sufficiently small) that

$$\|u\|_{L^{10}_{t,x}(I_j \times \mathbb{R}^3)} \lesssim 1$$

Summing this over each of the $O(|I|/\delta)$ intervals I_j we obtain the desired $L^{10}_{t,x}$ bound. The $\dot{S}^1$ bound then follows as in Theorem 4.8. □

The above corollary gives the desired contradiction to (161) when N_{max}/N_{min} is bounded, i.e. $N(t)$ stays in a bounded range.

8.17. **Third stage: Nonconcentration of energy.** Of course, any global well-posedness argument for (148) must eventually exclude a blowup scenario (self-similar or otherwise) where $N(t)$ goes to infinity in finite time, and indeed by Corollary 8.16 this is the only remaining possibility for a *minimal energy blowup solution*. Corollary 8.6 implies that in such a scenario the energy must almost entirely evacuate the frequencies near N_{min}, and instead concentrate at frequencies much larger than N_{min}. While this scenario is consistent with conservation of energy, it turns out to not be consistent with the time and frequency distribution of mass.

More specifically, we know there is a $t_{min} \in I_0$ so that for all $t \in I_0, N(t) \geq N(t_{min}) := N_{min} > 0$. By Corollary 8.6, at time t_{min} the solution has the bulk of its energy near the frequency N_{min}, and hence the medium frequencies at that time have mass bounded below by,

$$\|P_{c(\eta_0)N_{min} \leq \cdot \leq C(\eta_0)N_{min}} u(t_{min})\|_{L^2} \gtrsim c(\eta_0) N^{-1}_{min}. \tag{171}$$

The idea is to prove the following approximate mass conservation law for these high frequencies[35], which states that while some mass might slip to very low frequencies, it can't all do so.

[35] It is necessary to truncate to the high frequencies in order to exploit mass conservation because the low frequencies contain an unbounded amount of mass. This strategy of mollifying the solution in frequency space in order to exploit a conservation law that would otherwise be unbounded or useless is inspired by the "I-method" for sub-critical dispersive equations discussed in Lecture # 4.

Lemma 8.18 (Some mass freezes away from low frequencies). *Suppose u is a minimal energy blowup solution of* (148). *Then for all $t \in I_0$,*

$$\|P_{\geq \eta_4^{100} N_{min}} u(t)\|_{L^2} \gtrsim \eta_1. \tag{172}$$

Lemma 8.18 will quickly show that the evacuation scenario - wherein the solution cleanly concentrates energy to very high frequencies - cannot occur. Instead the solution always leaves a nontrivial amount of mass and energy behind at medium frequencies. This "littering" of the solution will serve (via Corollary 8.6) to keep $N(t)$ from escaping to infinity[36] and gives us,

Proposition 8.19 (Energy cannot evacuate from low frequencies). *For any minimal energy blowup solution of* (148) *we have*

$$N(t) \lesssim C(\eta_5) N_{min} \tag{173}$$

for all $t \in I_0$.

By combining Proposition 8.19 with Corollary 8.16, we encounter a contradiction to (161) which completes the proof of Theorem 7.3.

We conclude this lecture by summarizing the role of the parameters $\eta_i, i = 0, \ldots 5$ which have now all been introduced. The number η_1 represents the amount of potential energy that must be present at every time in a minimal energy blowup solution. (Proposition 8.7); it also represents the extent of concentration of energy (on the scale of $1/N(t)$) that must occur in physical space at every time in a minimal energy blowup solutions (Proposition 8.8). The number η_2 is introduced in Proposition 8.9, where $1/\eta_2$ represents the extent that there is localization (on the scale of $1/N(t)$) of energy in a minimal energy blowup solution. The number η_3 measures, on the scale of the quantity N_{min}, what we mean by "high frequency" when we say Proposition 8.12 is an interaction Morawetz estimate localized to high frequencies. The number η_4 measures the frequency (on the scale of N_{min}) below which the evolution can't move a certain portion (namely, η_1) of the L^2 mass. Finally, the number η_0 enters in Corollary 8.6 and various other points in the paper where we simply use its value as a small, universal constant.

[36] It is interesting to note that one must exploit conservation of energy, conservation of mass, *and* conservation of momentum (via the Morawetz inequality) in order to prevent blowup for the equation (148); the same phenomenon occurs in the previous arguments [13], [41] in the radial case, even though the details of those arguments are in many ways quite different to those here.

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