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Periodic Schroödinger equations in Hamiltonian form

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In these lectures I will summarize some old and recent results concerning different aspects of periodic Schrödinger equations viewed as infinite dimension Hamiltonian systems.

In the first lecture I will recall few facts about dispersive equations and nonlinear Schr´dingier equations in general, see Sections 1.1 and 1.2. In the second lecture, see Section 1.3, I will start with the classical Strichartz inequalities in one and two dimensions due to Bourgain. I will continue with some results of local well-posedness for certain nonlinear Schrödinger equations.

In the third lecture, see Section 1.4, I will elaborate on the growth in time of high order Sobolev norms for global smooth solutions. I will explain how the estimate of this growth could give some information on how the frequency profile of a certain wave solution could move from low to high frequencies while maintaining constant mass and energy (*forward cascade*.) I will present two results for the defocusing, cubic, periodic, two dimensional Schrödinger equation: the first is a polynomial upper bound in time for Sobolev norms of a global generic solution; the second is a weak growth result, namely that after fixing a small constant δ and a large one K, one can find a certain solution that at time zero is as small[1] as δ and at a certain time far in the future is as big as K.

In the forth lecture, see Section 1.5, I will introduce Gibbs measures associated to certain periodic nonlinear Schrödinger equations in one dimension. Here I will recall the result of Bourgain in which he proves invariance of these measures and uses this invariance in order to prove global well-posedness at a level in which conservation laws are not available. Of course in this case global well-posedness should be understood as an almost sure result.

In the fifth lecture, see Section 1.6, I will discuss the periodic derivative nonlinear Schrödinger (DNLS) equation. This is an integrable system but to prove that it is globally well-posed for rough data it is a enterprise. In order to

[1] In terms of a fixed Sobolev norm.

be able to use certain estimates one needs to apply a gauge transformation to the equation. Moreover for the gauged equation local well-posedness is know only on certain spaces that are of type l^p not necessarily $p = 2$, with respect to frequency variables. In fact when one wants to introduce a Gibbs measure, which is in turn related to the Gaussian measure defined on Sobolev spaces $H^s, s < \frac{1}{2}$, one needs to generalize the definition and take advantage of the more abstract Wiener theory. For this reason we introduce a weighted Wiener measure for the gauged equation and apply a generalization of Bourgain's argument to obtain again an almost surely global well-posedness result.

In the sixth lecture, see Section 1.7, I will show how a purely probabilistic argument will translate the almost surely global well-posedness for the gauged DNLS into a similar one for the original derivative nonlinear Schrödinger equation.

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1.1 Setting up the stage

The objects of study in these lectures are two initial value problems of Schrödinger type. The first is the semilinear Schrödinger (NLS) initial value problems (IVP)

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where $p > 1$, $u : \mathbb{R} \times \mathbb{T}^n \to \mathbb{C}$, and $\mathbb{T}^n$ is a n-dimensional torus[2]; the second is the derivative nonlinear Schrödinger (DNLS) initial value problem

$$\begin{cases} iu_t(x,t) + u_{xx}(x,t) = i\lambda \left(|u|^2(x,t)u(x,t)\right)_x \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{T}$. In both cases $t \in \mathbb{R}$ is the time variable and $\lambda = \pm 1$, and the sign will be determined later. Even in these relatively special case we will not be able to mention all the findings and results concerning the initial value problems (1.1) and (1.2) and for this we apologize in advance.

Schrödinger equations are classified as *dispersive* partial differential equations and the justification for this name comes from the fact that if no boundary conditions are imposed their solutions tend to be waves which spread out spatially. But what does this mean mathematically? A simple and complete mathematical characterization of the word *dispersion* is given to us for example by R. Palais in [52]. Although his definition is given for one dimensional

[2] Later we will distinguish between a rational and an irrational torus.

waves, the concept is expressed so clearly that it is probably a good idea to follow almost[3] literally his explanation: *"Let us [next] consider* linear *wave equations of the form*

$$u_t + P\left(\frac{\partial}{\partial x}\right) u = 0,$$

where P is polynomial. Recall that a solution $u(x,t)$, which Fourier transform is of the form $e^{i(kx-\omega t)}$, is called a plane-wave solution; k is called the wave number *(waves per unit of length) and ω the (angular)* frequency. *Rewriting this in the form $e^{ik(x-(\omega/k)t)}$, we recognize that this is a traveling wave of velocity $\frac{\omega}{k}$. If we substitute this $u(x,t)$ into our wave equation, we get a formula determining a unique frequency $\omega(k)$ associated to any wave number k, which we can write in the form*

$$\frac{\omega(k)}{k} = \frac{1}{ik} P(ik). \tag{1.3}$$

This is called the "dispersive relation" for this wave equation. Note that it expresses the velocity for the plane-wave solution with wave number k. For example, $P(\frac{\partial}{\partial x}) = c\frac{\partial}{\partial x}$ gives the linear advection equation $u_t + cu_x = 0$, which has the dispersion relation $\frac{\omega}{k} = c$, showing of course that all plane-wave solutions travel at the same velocity c, and we say that we have trivial dispersion in this case. On the other hand if we take $P\left(\frac{\partial}{\partial x}\right) = -\frac{i}{2}\left(\frac{\partial}{\partial x}\right)^2$, then our wave equation is $iu_t + \frac{1}{2}u_{xx} = 0$, which is the linear Schrödinger equation, and we have the non-trivial dispersion relation $\frac{\omega}{k} = \frac{k}{2}$. In this case, plane waves of large wave-number (and hence high frequency) are traveling much faster than low-frequency waves. The effect of this is to "broaden a wave packet". That is, suppose our initial condition is $u_0(x)$. We can use the Fourier transform[4] to write u_0 in the form

$$u_0(x) = \int \widehat{u_0}(k) e^{ikx}\, dk, \tag{1.4}$$

and then, by superposition, the solution to our wave equation will be

$$u(x,t) = \int \widehat{u_0}(k) e^{ik(x-(\omega(k)/k)t)}\, dk.$$

Suppose for example that our initial wave form is a highly peaked Gaussian. Then in the case of the linear advection equation all the Fourier modes travel together at the same speed and the Gaussian lump remains highly peaked over time. On the other hand, for the linearized Schrödinger equation the various Fourier modes all travel at different velocities, so after time they start canceling each other by destructive interference, and the original sharp Gaussian quickly broadens".

[3] R. Palais actually uses the Airy equation as an example, while we use the linear Schrödinger equation to be consistent with the topic of the lectures.

[4] In these lectures we will ignore the absolute constants that may appear in other definitions for the Fourier transform.

As one can imagine dispersive equations are proposed as descriptions of certain wave phenomena that occur in nature. But it turned out that some of these equations also appear in more abstract mathematical areas such as algebraic geometry [38], and certainly we are not in the position to discuss this beautiful part of mathematics here.

The interesting aspect of dispersive equations and Schrödinger equations in particular is that at later times, its solutions do not acquire pointwise extra smoothness, but only a weak smoothness on overage. In particular, since we will impose periodic boundary conditions, dispersion will be extremely weak and no even the weak smoothness on overage mentioned above will be available. All this will make our analysis more difficult, but also more interesting.

The questions we start addressing first are phenomenological. Assume that a profile of a wave is given at time $t = 0$, (initial data). Is it possible to prove that there exists a unique wave that "lives" for an interval of time $[0, T]$, that satisfies the equation, and that at time $t = 0$ has the assigned profile? What kind of properties does the wave have at later times? Does it "live" for all times or does it "blow up" in finite time?

Our intuition tells us that if we start with *nice* and *small* initial data, then all the questions above should be easier to answer. This is indeed often true. In general, in this case one can prove that the wave exists for all times, it is unique and its "size", measured taking into account the order of smoothness, can be controlled in a reasonable way. But what happens when we are not in this advantageous setting? These lecture notes are devoted to the understanding of how much of the above is still true when we consider *large* data and *long* interval of times when we are in a periodic setting. We will also introduce a more probabilistic approach to the problem when the data assigned are too rough to hope for a deterministic study of the problem.

To be able to give a rigorous setting for the study of the initial value problem in (1.1) and to avoid any confusion in the future, we need a strong mathematical definition for *well-posedness*. We consider the general initial value problem of type

$$\begin{cases} \partial_t u + P_m(\partial_{x_1}, \dots, \partial_{x_n})u + N(u, \partial_x^\alpha u) = 0, \\ u(x,0) = u_0(x), \qquad\qquad x \in \mathbb{R}^n \text{ or } x \in \mathbb{T}^n,\ t \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $m \in \mathbb{N}$, $P_m(\partial_{x_1}, \dots, \partial_{x_n})$ is a differential operator with constant coefficients of order m and $N(u, \partial_x^\alpha u)$ is the nonlinear part of the equation, that is a nonlinear function that depends on u and derivatives of u up to order $m-1$. The function $u_0(x)$ is the initial condition or initial profile, and most of the time is called initial data. Above, we pointed out the fact that finding a solution for an IVP strongly depends on the regularity one asks for the solution itself. So we first have to decide how we "measure" the regularity of a function. The most common way of doing so is to decide where the weak derivatives of the function "live". It is indeed time to recall the definition of Sobolev spaces[5]

Definition 1.1. *We say that a function* $f \in H^k(\mathbb{R}^n), k \in \mathbb{N}$ *if* f *and all its partial derivatives up to order* k *are in* L^2. *We recall that* $H^k(\mathbb{R}^n)$ *is a Banach space with the norm*

$$\|f\|_{H^k} = \sum_{|\alpha|=0}^{k} \|\partial_x^\alpha f\|_{L^2},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ *and* $|\alpha| = \sum_{i=1}^n \alpha_i$ *is its length.*

Remark 1.2. Because $\widehat{\partial_x^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$, it is easy to see that $f \in H^k(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2k} \, d\xi < \infty,$$

and moreover

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2k} \, d\xi \right)^{1/2} \sim \|f\|_{H^k}.$$

Then we can generalize our notion of Sobolev space and define $H^s(\mathbb{R}^n), s \in \mathbb{R}$ as the set of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s} \, d\xi < \infty.$$

Also $H^s(\mathbb{R}^n)$ is a Banach space with norm

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s} \, d\xi \right)^{1/2}.$$

Sometimes it is useful to use the *homogeneous* Sobolev space $\dot{H}^s(\mathbb{R}^n)$. This is the space of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s} \, d\xi < \infty.$$

Clearly all these observations and definitions can be made for Sobolev spaces in $\mathbb{T}^n$, except that in this case $\dot{H}^s(\mathbb{T}^n)$ and $H^s(\mathbb{T}^n)$ coincides.

We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^n)$ norm. We often need mixed norm spaces, so for example, we say that $f \in L^p_x L^q_t$ if $\|(\|f(x,t)\|_{L^q_t})\|_{L^p_x} < \infty$. Finally, for a fixed interval of time $[0, T]$ and a Banach space of functions Z, we denote with $C([0, T], Z)$ the space of continuous maps from $[0, T]$ to Z.

We are now ready to give a first definition of well-posedness. We will give a more refined one later in Subsection 1.2.4.

[5] In more sophisticated instances one replaces Sobolev spaces with different ones, like L^p spaces, Hölder spaces, and so on.

Definition 1.3. *We say that the IVP* (1.5) *is locally well-posed (l.w.p) in* H^s *if, given* $u_0 \in H^s$, *there exist* T, *a Banach space of functions* $X_T \subset C([-T,T];H^s)$ *and a unique* $u \in X_T$ *which solves*[6] (1.5). *Moreover we ask that there is continuity with respect to the initial data in the appropriate topology. We say that* (1.5) *is globally well-posed (g.w.p) in* H^s *if the definition above is satisfied in any interval of time* $[-T,T]$.

Remark 1.4. The intervals of time are symmetric about the origin because the problems that we study here are all time reversible (i.e. if $u(x,t)$ is a solution, then so is $-u(x,-t)$).

1.1.1 Notation

Throughout these notes we use C to denote various constants. If C depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C_{\|u_0\|_2}$ will depend on $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is an absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a+\varepsilon$ and $a-\varepsilon$, for some $0<\varepsilon \ll 1$.

Finally, since we will be making heavy use of Fourier transforms, we recall here that $\hat{f}$ will usually denote the Fourier transform of f with respect to the space variables and when there is no confusion we use the hat notation even when we take Fourier transform also with respect to the time variable. In general though we will use the notation $\tilde{u}$ if we want to emphasize that we take the Fourier transform of a function $u(t,x)$ in both space and time variables.

1.2 The Nonlinear Schrödinger Equation (NLS). An introduction

We consider the (NLS) IVP (1.1) and for now we formally talk about the solution $u(x,t)$ as an object that exists, is smooth, etc. Of course to be able to use whatever we say here later we will need to work on making this formal assumption true!

Given an equation it is always a good idea to read as much as possible out of it. So one should always ask what are the rigid constraints that an equation imposes on its solutions a-priori. Here we will look at conservation laws (in this case integrals involving the solution that are independent of time), scaling and invariances that a solution to (1.1) can be subject to. It is true though that in describing these important features of the equation one often has to

[6] The notion of solution needs to be explained since we may be dealing with very low regularity and a classical notion may not be suitable. In general in our context a solution is a function that satisfies the integral equation equivalent to the IVP at hand through the Duhamel principle.

recall some basic principles/quantities coming from physics like conservation of mass, energy and momentum, the notion of density, interaction of particles, resonance, etc.

1.2.1 Conservation laws

A simple way to interpret physically the function $u(x,t)$ solving a Schrödinger equation is to think about $|u(x,t)|^2$ as the particle density at place x and at time t. Then it shouldn't come as a surprise that the density, momentum and energy are conserved in time. More precisely if we introduce the *pseudo-stress-energy tensor* $T_{\alpha,\beta}$ for $\alpha, \beta = 0, 1, ..., n$

$$T_{00} = |u|^2 \quad \text{(mass density)} \tag{1.6}$$

$$T_{0j} = T_{j0} = Im(\bar{u}\partial_{x_j} u) \quad \text{(momentum density)} \tag{1.7}$$

$$T_{jk} = Re(\partial_{x_j} u \overline{\partial_{x_k} u}) - \frac{1}{4}\delta_{j,k}\Delta(|u|^2) + \lambda\frac{p-1}{p+1}\delta_{jk}|u|^{p+1} \quad \text{(stress tensor)} \tag{1.8}$$

then by using the equation one can show that

$$\partial_t T_{00} + \partial_{x_j} T_{0j} = 0 \quad \text{and} \quad \partial_t T_{j0} + \partial_{x_k} T_{jk} = 0 \tag{1.9}$$

for all $j, k = 1, ..., n$, for more details see also [61].

The conservation laws summarized in (1.9) are said to be *local* in the sense that they hold pointwise in the physical space. Clearly by integrating in space and assuming appropriate boundary conditions for u one also has the conserved integrals

$$m(t) = \int T_{00}(x,t)\,dx = \int |u|^2(x,t)\,dx \quad \text{(mass)} \tag{1.10}$$

$$p_j(t) = -\int T_{0j}(x,t) = -\int Im(\bar{u}\partial_{x_j} u)\,dx \quad \text{(momentum).} \tag{1.11}$$

We observe here that the stress tensor in (1.8) is not conserved, but it plays an important role in some "sophisticated" monotonicity formulas involving the solution u.

The energy $E(u(t))$ is defined as

$$E(u(t)) = K(t) + P(t)$$

the sum of kinetic and potential energy. In our case

$$K(t) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx \quad \text{and} \quad P(t) = \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx$$

and hence

$$E(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx. \tag{1.12}$$

It turned out that $E(u(t))$ is actually the Hamiltonian of the system and hence it is conserved. One can also prove that $\frac{d}{dt}E(u(t)) = 0$ directly by using the equation. We immediately observe that now the sign of λ in (1.12) plays a very important role since by picking $\lambda = -1$ one can produce a negative energy.

1.2.2 The periodic NLS as an Hamiltonian system

We mentioned above that the energy $E(u(t))$ in (1.12) is the Hamiltonian for the system. In fact the periodic NLS (1.1) can be viewed as an infinite dimensional Hamiltonian system. Let's recall that

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}(-\frac{\partial H}{\partial q_i}) = 0.$$

If we define $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Going back to (1.1) and (1.12) one can set $E(u) = H(u, \bar{u})$ and check that the equation (1.1) is equivalent to

$$\dot{u} = i\frac{\partial H(u,\bar{u})}{\partial \bar{u}}$$

and one can think of u as the infinite dimension vector given by its Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}^n}$. Once this analogy has ben established one may want to study how far one can push it. In particular is a result such as Gromov's non-squeezing theorem [6, 25, 41] true hereWe will not address this particular question in these notes.? Can one use the Hamiltonian structure to define an invariant measure [5, 43] in the infinite dimensional space given by the vectors of Fourier coefficients? We will consider this last question in Section 1.5.

Remark 1.5. For the problem (1.2) one can similarly prove invariance of certain integrals. In fact, it is know that the system is integrable, that is it admits an infinite number of conservation laws [35]. In particular, the energy and Hamiltonian:

$$\text{Energy:} \quad E(u) = \int |u_x|^2\,dx + \frac{3}{2}\text{Im}\int u^2\overline{u}\,\overline{u_x}\,dx + \frac{1}{2}\int |u|^6\,dx. \tag{1.13}$$

$$\text{Hamiltonian:} \quad H(u) = \text{Im}\int u\overline{u}_x\,dx + \frac{1}{2}\int |u|^4\,dx, \tag{1.14}$$

as well as the mass are conserved.

As a consequence, the DNLS equation (1.2) can also be viewed as an infinite dimension Hamiltonian system. We will concentrate on this more in Section 1.5.

As we will see later, having an a-priori control in time of an energy like in (1.12) when $\lambda = 1$ is an essential tool in order to prove that a solution exists for all times. But it is also true that often this is not sufficient. This is indeed the case when the problem is *critical*[7].

1.2.3 Scaling Symmetry

A good tool to at least assess how difficult the question of well-posedness may be for a certain evolution equation is scaling symmetry.

If u solve the IVP (1.1) then

$$u_\mu(x,t) = \mu^{-\frac{2}{p-1}} u\left(\frac{t}{\mu^2}, \frac{x}{\mu},\right) \quad \text{and} \quad u_{\mu,0}(x) = \mu^{-\frac{2}{p-1}} u\left(\frac{x}{\mu},\right) \tag{1.15}$$

solves the IVP for any $\mu \in \mathbb{R}$. We now show how this can be used to understand for which non-linerity (or for which $p > 1$) the problem of well-posedness is most difficult to address.

If we compute $\|u_{\mu,0}\|_{\dot{H}^s}$ we see that

$$\|u_{\mu,0}\|_{\dot{H}^s} \sim \mu^{-s+s_c} \|u_0\|_{\dot{H}^s}, \tag{1.16}$$

where

$$s_c = \frac{n}{2} - \frac{2}{p-1}.$$

From (1.16) it is clear that if we take $\mu \to +\infty$ then

1. if $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the interval of time is made longer: our intuition says that this is the best possible setting for well-posedness,
2. if $s = s_c$ (**critical case**) the norm is invariant while the interval of time is made longer. This tells us that rescaling is not improving the situation: nothing can be made *small*.
3. if $s < s_c$ (**super-critical case**) the norm grows as the time interval gets longer. Scaling is obviously against us.

In order to provide a better intuition for scaling we present an informal argument as in [61] which relates the dispersive part of the solution, Δu, with the nonlinear part, $|u|^{p-1}u$. Let's consider a special type of initial wave u_0. We want u_0 such that its support in Fourier space is localized at a large frequency $N \gg 1$, and contained in a ball of radius $1/N$ and its amplitude is A. Here

[7] The notion of criticality will be introduced below.

we are making the assumption that scaling is the only symmetry that could interfere with a behavior that goes from linear to nonlinear, in fact in general this is not the only one. We have

$$\|u_0\|_{L^2} \sim AN^{-n/2}, \quad \|u_0\|_{\dot{H}^s} \sim AN^{s-n/2}.$$

If we want $\|u_0\|_{\dot{H}^s}$ small then we need to ask that $A \ll N^{n/2-s}$. Under this restriction we want to compare the liner term Δu with the nonlinear part $|u|^{p-1}u$:

$$|\Delta u| \sim AN^2 \quad \text{while} \quad |u|^p \sim A^p.$$

From here if $AN^2 \gg A^p$ we believe that the linear behavior would win, alternatively the nonlinear one would. Putting everything together we have that

$$A^{p-1} \ll N^2 \quad \text{and} \quad A \ll N^{n/2-s} \Longrightarrow s > s_c \quad \text{(more linear)} \tag{1.17}$$

$$A^{p-1} \gg N^2 \quad \text{and} \quad A \gg N^{n/2-s} \Longrightarrow s < s_c \quad \text{(more nonlinear)}. \tag{1.18}$$

As announced at the start of this argument, this so called "scaling argument" should only be used as a guideline since in delivering it we make a purely formal calculation. On the other hand, in some cases ill-posedness results below critical exponent have been obtained (see for example [18, 19]).

1.2.4 Definition of well-posedness

We conclude this lecture by giving the precise definition of local and global well-posedness for the IVP (1.1).

Definition 1.6 (Well-posedness). *We say that the IVP* (1.1) *is* locally well-posed (l.w.p) *in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$ there exist a time T and a Banach space of functions $X \subset L^\infty([-T,T], H^s(\mathbb{R}^n))$ such that for each initial data $u_0 \in B$ there exists a unique solution $u \in X \cap C([-T,T], H^s(\mathbb{R}^n))$ for the integral equation*

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')(|u|^{p-1}u)(t')\,dt'. \tag{1.19}$$

Furthermore the map $u_0 \to u$, that associates the solution to the initial datum, is continuous as a map from H^s into $C([-T,T], H^s(\mathbb{R}^n))$. If uniqueness is obtained in $C([-T,T], H^s(\mathbb{R}^n))$, then we say that local well-posedness is unconditional.

If this hold for all $T \in \mathbb{R}$ then we say that the IVP is *globally well-posed (g.w.p).*

Remark 1.7. Our notion of global well-posedness does not require that the norm $\|u(t)\|_{H^s(\mathbb{R}^n)}$ remains uniformly bounded in time. In fact, unless $s = 0, 1$ and one can use the conservation of mass or energy, it is not a triviality to show such a uniform bound. Such bounds can be obtained as a consequence of scattering, when scattering is available and in general, it is a question related to *weak turbulence theory* and we will address it more in details in Section 1.4.

1.3 Periodic Strichartz estimates

In this section we introduce some of the most important estimates relative to the linear Schrödinger IVP

$$\begin{cases} iv_t + \Delta v = 0, \\ v(x,0) = u_0(x). \end{cases} \tag{1.20}$$

It is important to understand the solution v of (1.20), which we will denote with $v(x,t) = S(t)u_0(x)$, as much as possible. In fact by the Duhamel principle one can write the solution of the associated forced or nonlinear problem

$$\begin{cases} iu_t + \Delta u = F(u), \\ u(x,0) = u_0(x) \end{cases} \tag{1.21}$$

as

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')F(u(t'))\,dt'. \tag{1.22}$$

The solution of the linear problem (1.20) is easily computable by taking Fourier transform. In fact by fixing the frequency ξ problem (1.20) transforms into the ODE

$$\begin{cases} i\hat{v}_t(t,\xi) - |\xi|^2\hat{v}(t,\xi) = 0, \\ \hat{v}(\xi,0) = \hat{u}_0(\xi) \end{cases} \tag{1.23}$$

and we can write its solution as

$$\hat{v}(t,\xi) = e^{-i|\xi|^2 t}\hat{u}_0(\xi).$$

In general the solution $v(t,x)$ above is denoted by $S(t)u_0$, where $S(t)$ is called the Schrödinger group. We observe that what we just did works both in $\mathbb{R}^n$ and $\mathbb{T}^n$.

1.3.1 Strichartz estimates in $\mathbb{R}^n$

If we define, in the distributional sense,

$$K_t(x) = \frac{1}{(\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}},$$

then we have

$$S(t)u_0(x) = e^{it\Delta}u_0(x) = u_0 \star K_t(x) = \frac{1}{(\pi i t)^{n/2}}\int e^{i\frac{|x-y|^2}{2t}} u_0(y)\,dy. \tag{1.24}$$

As mentioned already

$$\widehat{S(t)u_0}(\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi), \tag{1.25}$$

and this last equation can be interpreted as the adjoint of the Fourier transform restricted on the paraboloid $P = \{(\xi, |\xi|^2) \ \text{ for } \xi \in \mathbb{R}^n\}$. This remark, strictly linked to (1.24) and (1.25), can be used to prove a variety of very deep estimates for $S(t)u_0$, see for example [58, 17]. From (1.24) we immediately have the so called *Dispersive Estimate*

$$\|S(t)u_0\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\|u_0\|_{L^1}. \tag{1.26}$$

From (1.25) we have the conservation of the homogeneous Sobolev norms[8]

$$\|S(t)u_0\|_{\dot{H}^s} = \|u_0\|_{\dot{H}^s}, \tag{1.27}$$

for all $s \in \mathbb{R}$. Interpolating (1.26) with (1.27) when $s = 0$ and using a so called TT^* argument one can prove the famous Strichartz estimates (see [17], [39], and [61] for some concise proofs, and [17] for a complete list of authors who contributed to the final version of the theorem below):

Theorem 1.8. *[Strichartz Estimates in $\mathbb{R}^n$] Fix $n \geq 1$. We call a pair (q, r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q, r, n) \neq (2, \infty, 2)$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the homogeneous Strichartz estimate*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{1.28}$$

and the inhomogeneous Strichartz estimate

$$\left\|\int_0^t S(t-t')F(t')\,dt'\right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{1.29}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

1.3.2 Strichartz estimates in $\mathbb{T}^n$

In this section we will see how essential is the assumption that $\mathbb{T}^n$ is a rational torus[9] in order to be able to prove sharp Strichartz estimates. The conjecture is that for irrational tori one should be able to prove similar estimates, if not better in some cases, but for now the best available results are due to Bourgain in [11, 12]. In a sense irrational tori should generate some sort of

[8] We will see later that the L^2 norm is conserved also for the nonlinear problem (1.1).

[9] For us a torus is irrational if there are at least two coordinates for which the ratio of their periods is irrational.

weak dispersion since the reflections of the wave solutions through periodic boundary conditions, with periods irrational with respect to each other, should interact less in the nonlinearity. As for now there are no results of this type in the literature.

Assume that $c_i > 0$, $i = 1, ..., n$ are the periods with respect to each coordinate. In the periodic case one cannot expect the range of admissible pairs (q, r) as in Theorem 1.8. We concentrate on the pairs $q = r$, that is $q = \frac{2(n+2)}{n}$. There is the following conjecture:

Conjecture 1.9. Assume that $\mathbb{T}^n$ is a rational torus and the support of $\hat{\phi}_N$ is in the ball $B_N(0) = \{|n| \lesssim N\}$. Write

$$S(t)\phi_N(x) = \sum_{k\in\mathbb{Z}^n, |k|\sim N} a_k e^{i(\langle x,k\rangle - \gamma(k)t)},$$

where (a_k) are the Fourier coefficients of ϕ_N and

$$\gamma(k) = \sum_{i=1}^{n} c_i k_i^2. \tag{1.30}$$

If the torus is rational we can assume without loss of generality that $c_i \in \mathbb{N}$. Then for $q \geq 2$

$$\|S(t)\phi_N\|_{L_t^q L_x^q([0,1]\times\mathbb{T}^n)} \lesssim C_q \|\phi_N\|_{L_x^2(\mathbb{T}^n)} \quad \text{if } q < \frac{2(n+2)}{n} \tag{1.31}$$

$$\|S(t)\phi_N\|_{L_t^q L_x^q([0,1]\times\mathbb{T}^n)} \ll N^{\epsilon} \|\phi_N\|_{L_x^2(\mathbb{T}^n)} \quad \text{if } q = \frac{2(n+2)}{n} \tag{1.32}$$

$$\|S(t)\phi_N\|_{L_t^q L_x^q([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\frac{n}{2}-\frac{n+2}{q}} \|\phi_N\|_{L_x^2(\mathbb{T}^n)} \quad \text{if } q < \frac{2(n+2)}{n} \tag{1.33}$$

For a partial resolution of the conjecture see [4]. We present Bourgain's argument for $n = 2$, $q = 4$ below to show how the rationality of the torus comes into play.

Proof. In this proof we restrict further to the case when $c_i = 1$ for $i = 1, 2$. Then

$$\begin{aligned} &\left\| \sum_{|k|\leq N} a_k e^{i(\langle x,k\rangle - |n|^2 t)} \right\|^4_{L^4([0,1]\times\mathbb{T}^2)} \\ &= \left\| \left[\sum_{|k|\leq N} a_k e^{i(\langle x,k\rangle - |k|^2 t)} \right]^2 \right\|^2_{L^2([0,1]\times\mathbb{T}^2)} = \sum_{k,m} |b_{k,m}|^2, \end{aligned}$$

where

$$b_{k,m} = \sum_{k=k_1+k_2; m=|k_1|^2+|k_2|^2, |k_i|\leq N, i=1,2} a_{k_1} a_{k_2}$$

since

$$\left[\sum_{|k|\leq N} a_k e^{i(\langle x,k\rangle - |n|^2 t)}\right]^2 = \sum_{|n_1|\leq N, |n_2|\leq N} a_{k_1} a_{k_2} e^{i(\langle x,(k_1+k_2)\rangle - (|k_1|^2+|k_2|^2)t)}$$
$$= \sum_{k,m} b_{k,m} e^{i(\langle x,k\rangle + mt)}.$$

Now it is easy to see that

$$\|S(t)\phi_N\|^4_{L^4_t L^4_x([0,1]\times\mathbb{T}^2)} \sim \sum_{k,m} |b_{k,m}|^2 \lesssim \sup_{|k|\lesssim N, |m|\lesssim N^2} \# M(k,m) \|(a_k)\|^4_{l^2}, \tag{1.34}$$

where

$$\# M(k,m) = \#\{(k_1 \in \mathbb{Z}^2 \ / \ 2m - |k|^2 = |k-2k_1|^2\} = \#\{(z \in \mathbb{Z}^2 \ / \ 2m - |k|^2 = |z|^2\}.$$

If $2m - |k|^2 < 0$ there are no points in $M(k,m)$, and if $R^2 := 2m - |k|^2 \geq 0$, there are at most $\exp C \frac{\log R}{\log\log R}$ many points on the circle of radius R [32], and since $R^2 \leq N^2$, using (1.34) we obtain

$$\|S(t)\phi_N\|_{L^4_t L^4_x([0,1]\times\mathbb{T}^2)} \lesssim N^{\epsilon} \|\phi_N\|_{L^2}, \tag{1.35}$$

for all $\epsilon > 0$.

Remark 1.10. Thanks to a very precise translation invariance in the frequency space for $S(t)u$, estimate (1.35) holds also when the support of ϕ_N is on a ball of radius N centered in an arbitrary point $z_0 \in \mathbb{Z}^2$.

In order to set up a fixed point theorem to prove well-posedness one defines $X^{s,b}$ spaces, introduced in this context by Bourgain [4]. The norms in these spaces are defined for $s, b \in \mathbb{R}$ as:

$$\|u\|_{X^{s,b}(\mathbb{T}^2\times\mathbb{R})} := \Big(\sum_{n\in\mathbb{Z}^2} \int_{\mathbb{R}} |\widetilde{u}(n,\tau)|^2 \langle n\rangle^{2s} \langle \tau + |n|^2\rangle^{2b} d\tau\Big)^{\frac{1}{2}},$$

One can immediately see that these spaces are measuring the regularity of a function with respect to certain parabolic coordinates, this to reflect the fact that linear Schrödinger solutions live on parabolas. Having defined the spaces one wants to relate their norms to certain $L^q_t L^p_x$ norms that are typical of Strichartz estimates as proved above in a special case. A key estimate, proved in [4], is

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}}. \tag{1.36}$$

This is proved by viewing u as sum of components supported on paraboloids that are at distance one from each other, using (1.35) on each of them and then reassembling the estimates using the weight $\langle \tau + |n|^2 \rangle^{2b}$. An additional estimate is:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{4}+}}. \tag{1.37}$$

The estimate (1.37) is a consequence of the following lemma [9].

Lemma 1.11. *Suppose that Q is a ball in $\mathbb{Z}^2$ of radius N and center z_0. Suppose that u satisfies $\operatorname{supp} \widehat{u} \subseteq Q$. Then*

$$\|u\|_{L^4_{t,x}} \lesssim N^{\frac{1}{2}} \|u\|_{X^{0,\frac{1}{4}+}}. \tag{1.38}$$

Lemma 1.11 is proved in [9] by using Hausdorff-Young and Hölder's inequalities. We omit the details. We can now interpolate between (1.36) and (1.37) to deduce:

Lemma 1.12. *Suppose that u is as in the assumptions of Lemma 1.11, and suppose that $b_1, s_1 \in \mathbb{R}$ satisfy $\frac{1}{4} < b_1 < \frac{1}{2}+, s_1 > 1 - 2b_1$. Then*

$$\|u\|_{L^4_{t,x}} \lesssim N^{s_1} \|u\|_{X^{0,b_1}}. \tag{1.39}$$

Lemma 1.12 can then be used to prove local well-posedness for the cubic NLS in $\mathbb{T}^2$ in H^s, $s > 0$. One in fact can set up a fixed point argument in the space $X^{s,b}$, $s > 0$, $b \sim \frac{1}{2}$. The key point is that the problem at hand has a cubic nonlinearity which by duality forces us to consider a product of four functions in L^1. This translates into estimating L^4 norms which via (1.39) are related back to the space $X^{s,b}$. In the proof one shows that the interval of time $[-T, T]$ suitable for a fixed point argument is such that

$$T \sim \|u_0\|_{H^s}^{-\alpha}, \tag{1.40}$$

for some $\alpha > 0$. As a consequence, the defocusing, cubic, periodic NLS problem (1.1) can be proved to be globally well-posed in $H^s, s \geq 1$ thanks to (1.40) and the conservation of the Hamiltonian (1.12). See [4, 9]. We summarize all this in the theorem below.

Theorem 1.13. *The initial value problem* (1.1)*, where $p = 4$ and $\lambda = \pm 1$ is locally well-posed in $H^s, s > 0$. If $\lambda = -1$ (defocusing case) the problem is globally well-posed for $s \geq 1$.*

It should be remarked that one can extend global well-posedness below $s = 1$ by using the *I-method* [29]. We will not address this here.

1.4 Growth of Sobolev norms

We consider the cubic, defocusing, periodic (rational) NLS initial value problem:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \; x \in \mathbb{T}^2 \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}^2), \; s > 1. \end{cases} \tag{1.41}$$

As mentioned more in general in (1.10) and (1.12), this problem has the following conserved quantities:

$$\text{Mass} \quad M(u(t)) := \int |u(x,t)|^2 dx,$$

$$\text{Energy} \quad E(u(t)) := \frac{1}{2} \int |\nabla u(x,t)|^2 dx + \frac{1}{4} \int (|u|^2)(x,t) |u(x,t)|^2 dx.$$

From Theorem 1.13 we know that (1.41) is globally well-posed in $H^s, s \geq 1$. Hence, it makes sense to analyze the behavior of $\|u(t)\|_{H^s}$. But as we will discuss later this estimate is related to an important physical phenomenon which will be introduced in Subsection 1.4.1 below.

The main result of this section is a polynomial bound stated below.

Theorem 1.14 (Bound for the defocusing cubic NLS on $\mathbb{T}^2$; [55]). *Let u be the global solution of* (1.41) *on $\mathbb{T}^2$ and let $s > 1$. Then, there exists a function C_s, continuous on $H^1(\mathbb{T}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C_s(u_0)(1 + |t|)^{s+} \|u_0\|_{H^s(\mathbb{T}^2)}. \tag{1.42}$$

Remark 1.15. Let us observe that, when s is an integer, or when u_0 is smooth, essentially the same bound as in Theorem 1.14 was proved by using different techniques in the work of Zhong [67]. The reason why one uses the fact that s is an integer is because one wants to use exact formulae for the (Fractional) Leibniz Rule for D^s. By using an exact Leibniz Rule, one sees that certain terms which are difficult to estimate are in fact equal to zero.

Remark 1.16. Let us note that, if we consider the spatial domain to be $\mathbb{R}^2$, one can obtain uniform bounds on $\|u(t)\|_{H^s}$ for solutions $u(t)$ of the defocusing cubic NLS by the recent scattering result of Dodson [30].

1.4.1 Motivation for the problem and previously known results

The growth of high Sobolev norms has a physical interpretation in the context of the *Low-to-High frequency cascade.* In other words, we see that $\|u(t)\|_{H^s}$ weighs the higher frequencies more as s becomes larger, and hence its growth gives us a quantitative estimate for how much of the support of $|\widehat{u}|^2$ has transferred from the low to the high frequencies. This sort of problem also goes under the name *weak turbulence* [2, 3, 65]. By local well-posedness theory discussed at the end of Section 1.3, it can be observed that there exist $C, \tau_0 > 0$, depending only on the initial data u_0 such that for all t:

$$\|u(t + \tau_0)\|_{H^s} \leq C \|u(t)\|_{H^s}. \tag{1.43}$$

Iterating (1.43) yields the exponential bound:

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2|t|}, \tag{1.44}$$

where $C_1, C_2 > 0$ again depend only on u_0.

For a wide class of nonlinear dispersive equations, the analogue of (1.44) can be improved to a polynomial bound, as long as we take $s \in \mathbb{N}$, or if we consider sufficiently smooth initial data. This observation was first made in the work of Bourgain [7], and was continued in the work of Staffilani [56, 57].

The crucial step in the mentioned works was to improve the iteration bound (1.43) to:

$$\|u(t+\tau_0)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-r}. \tag{1.45}$$

As before, $C, \tau_0 > 0$ depend only on u_0. In this bound, $r \in (0,1)$ satisfies $r \sim \frac{1}{s}$. One can show that (1.45) implies that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(u_0)(1+|t|)^{\frac{1}{r}}. \tag{1.46}$$

In [7], (1.45) was obtained by using the *Fourier multiplier method*. In [56, 57], the iteration bound was obtained by using multilinear estimates in $X^{s,b}$-spaces. Similar estimates were used in the work of Kenig-Ponce-Vega [40] in the study of well-posedness theory. The key was to use a multilinear estimate in an $X^{s,b}$-space with negative first index. Such a bound was then used as a smoothing estimate. A slightly different approach, based on the analysis in the work of Burq-Gérard-Tzvetkov [13], is used to obtain (1.45) in the context of compact Riemannian manifolds in the work of Catoire-Wang [20], and Zhong [67].

In the case of the linear Schrödinger equation with potential on $\mathbb{T}^d$, better results are known. In [10], Bourgain studies the equation:

$$iu_t + \Delta u = Vu. \tag{1.47}$$

The potential V is taken to be jointly smooth in x and t with uniformly bounded partial derivatives with respect to both of the variables. It is shown that solutions to (1.47) satisfy for all $\epsilon > 0$ and all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C_{s,u_0,\epsilon}(1+|t|)^{\epsilon}. \tag{1.48}$$

The proof of (1.48) is based on separation properties of the eigenvalues of the Laplace operator on $\mathbb{T}^d$.

Recently, a new proof of (1.48) was given in the work of Delort [28]. The argument given in this paper is based on an iterative change of variable. In addition to recovering the result (1.48) on any d-dimensional torus, the same bound is proved for the linear Schrödinger equation on any Zoll manifold, i.e. on any compact manifold whose geodesic flow is periodic. So far, it is an open problem to adapt any of these techniques to obtain bounds like (1.48) for nonlinear equations.

We finally mention that the problem of Sobolev norm growth was also recently studied in [27], but in the sense of bounding the growth from below. In this paper, the authors exhibit the existence of smooth solutions of the cubic defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$, whose H^s norm is arbitrarily small at time zero, and is arbitrarily large at some large finite time. This will be the content of Section 1.4.3.

1.4.2 Polynomial bounds and the upside-down I-method

The main idea in the proof of Theorem 1.14 is to define $\mathcal{D}$ to be an *upside-down I-operator*. This operator is defined as a Fourier multiplier operator. By construction, we will be able to relate $\|u(t)\|_{H^s}$ to $\|\mathcal{D}u(t)\|_{L^2}$, so we consider the growth of the latter quantity. Following the ideas of the construction of the standard *I-operator*, as defined by Colliander, Keel, Staffilani, Takaoka, and Tao [21, 23, 24], our goal is to show that the quantity $\|\mathcal{D}u(t)\|^2_{L^2}$ is *slowly varying*. This is done by applying a Littlewood-Paley decomposition and summing an appropriate geometric series. Let us remark that a similar technique was applied in the low-regularity context in [23].

We will use *higher modified energies*, i.e. quantities obtained from $\|\mathcal{D}u(t)\|^2_{L^2}$ by adding an appropriate multilinear correction. In this way, we will obtain $E^2(u(t)) \sim \|\mathcal{D}u(t)\|^2_{L^2}$, which is even more slowly varying. Due to a more complicated resonance phenomenon in two dimensions, the construction of E^2 is going to be more involved than it was in [53], where the 1D problem was considered.

It is now time to give a precise definition of the operator $\mathcal{D}$. Suppose $N > 1$ is given. Let $\theta : \mathbb{Z}^2 \to \mathbb{R}$ be given by:

$$\theta(n) := \begin{cases} \left(\frac{|n|}{N}\right)^s, \text{if } |n| \geq N \\ 1, \text{ if } |n| \leq N \end{cases} \tag{1.49}$$

Then, if $f : \mathbb{T}^2 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(n) := \theta(n)\hat{f}(n). \tag{1.50}$$

We observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s \|\mathcal{D}f\|_{L^2}. \tag{1.51}$$

Our goal is to then estimate $\|\mathcal{D}u(t)\|_{L^2}$, from which we can estimate $\|u(t)\|_{H^s}$ by (1.51). In order to do this, we first need to have good local-in-time bounds.

Let u denote the global solution to (1.41) on $\mathbb{T}^2$. One then has:

Proposition 1.17. *(Local-in-time bounds for cubic defocusing NLS on $\mathbb{T}^2$) There exist $\delta = \delta(s, E(u_0), M(u_0)), C = C(s, E(u_0), M(u_0)) > 0$, which are* *continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{C}$ such that:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}. \tag{1.52}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(u_0), M(u_0)). \tag{1.53}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(u_0), M(u_0))\|\mathcal{D}u(t_0)\|_{L^2}. \tag{1.54}$$

The proof of Proposition 1.17 is based on a fixed-point argument. We need to use the estimates in $X^{s,b}$ spaces mentioned above in order to show that we obtain a contraction.

Remark 1.18. Although our statements concern functions which are only assumed to belong to $H^s(\mathbb{T}^2)$, with some fixed s, we can work with smooth functions and deduce the general result from the following approximation lemma.

Lemma 1.19. *(Approximation Lemma for the cubic NLS on $\mathbb{T}^2$) If u satisfies:*

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \\ u(x,0) = u_0(x). \end{cases} \tag{1.55}$$

and if the sequence $(u^{(n)})$ satisfies:

$$\begin{cases} iu_t^{(n)} + \Delta u^{(n)} = |u^{(n)}|^2 u^{(n)}, \\ u^{(n)}(x,0) = u_{0,n}(x). \end{cases} \tag{1.56}$$

where $u_{0,n} \in C^\infty(\mathbb{T}^2)$ and $u_{0,n} \xrightarrow{H^s} u_0, \quad s > 0$, then, one has for all t:

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

It is crucial that none of our estimates depend on higher Sobolev norms than H^s, which allows us to argue by density.

A higher modified energy and an iteration bound.

Let us give some useful notation for multilinear expressions, which can also be found in [21, 22]. For $k \geq 2$, an even integer, we define the hyperplane:

$$\Gamma_k := \{(n_1, \ldots, n_k) \in (\mathbb{Z}^2)^k : n_1 + \cdots + n_k = 0\},$$

endowed with the measure $\delta(n_1 + \cdots + n_k)$.

Given a function $M_k = M_k(n_1, \ldots, n_k)$ on Γ_k, i.e. a *k-multiplier*, one defines the *k-linear functional* $\lambda_k(M_k; f_1, \ldots, f_k)$ by:

$$\lambda_k(M_k; f_1, \dots, f_k) := \int_{\Gamma_k} M_k(n_1, \dots, n_k) \prod_{j=1}^{k} \widehat{f_j}(n_j).$$

As in [21], we adopt the notation:

$$\lambda_k(M_k; f) := \lambda_k(M_k; f, \bar{f}, \dots, f, \bar{f}). \tag{1.57}$$

We will also sometimes write $n_{ij\dots k}$ for $n_i + n_j + \cdots + n_k$.

We define the following *modified energy*:

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

By a calculation, we obtain that for some $c \in \mathbb{R}$, one has:

$$\frac{d}{dt}E^1(u(t)) = ic \sum_{\Gamma_4} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - ((\theta(n_4))^2\big) \times \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4). \tag{1.58}$$

One would like to show that this increment is small like $N^{-\alpha}$ for some $\alpha > 0$, but this is not true. We move then to a *higher modified energy*:

$$E^2(u) := E^1(u) + \lambda_4(M_4; u). \tag{1.59}$$

The modified energy E^2 is obtained by adding a "multilinear correction", the quantity M_4 determined later, to the modified energy E^1 we considered above. In order to find $\frac{d}{dt}E^2(u)$, we need to find $\frac{d}{dt}\lambda_4(M_4; u)$. If we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) = -i\lambda_4(M_4(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2); u) \tag{1.60}$$
$$-i \sum_{\Gamma_6} \big[M_4(n_{123}, n_4, n_5, n_6) - M_4(n_1, n_{234}, n_5, n_6) + M_4(n_1, n_2, n_{345}, n_6) - M_4(n_1, n_2, n_3, n_{456})\big] \times \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6).$$

We can compute that for $(n_1, n_2, n_3, n_4) \in \Gamma_4$, one has:

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 2n_{12} \cdot n_{14}. \tag{1.61}$$

We notice that (1.61) vanishes not only when $n_{12} = n_{14} = 0$, but also when n_{12} and n_{14} are orthogonal. Hence, on Γ_4, it is possible for $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2$ to vanish, but for $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ to be non-zero. Consequently, unlike in the 1D setting [53, 54], we can't cancel the whole quadrilinear term in (1.58). We remedy this by canceling the *non-resonant part* of the quadrilinear term. A similar technique was used in [26].

There, it was given the name *resonant decomposition*. More precisely, given $\beta_0 \ll 1$, which we determine later, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(n_1,n_2,n_3,n_4) \in \Gamma_4; n_{12}, n_{14} \neq 0, |cos\angle(n_{12},n_{14})| > \beta_0\} \tag{1.62}$$

and the set Ω_r of *resonant* frequencies Ω_r is defined to be its complement in Γ_4.

We now define the multiplier M_4 by:

$$M_4(n_1,n_2,n_3,n_4) := \begin{cases} c\frac{((\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2)}{|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2}, \text{in } \Omega_{nr} \\ 0, \text{ in } \Omega_r. \end{cases} \tag{1.63}$$

Next we define the multiplier M_6 on Γ_6 by:

$$M_6(n_1,n_2,n_3,n_4,n_5,n_6) := M_4(n_{123},n_4,n_5,n_6) - M_4(n_1,n_{234},n_5,n_6) + \\ + M_4(n_1,n_2,n_{345},n_6) - M_4(n_1,n_2,n_3,n_{456}). \tag{1.64}$$

We now use (1.58) and (1.60), and the construction of M_4 and M_6 to deduce that:

$$\frac{d}{dt}E^2(u) = \sum_{\Gamma_4, |cos\angle(n_{12},n_{14})| \leq \beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) \times \\ \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) \\ + \sum_{\Gamma_6} M_6(n_1,n_2,n_3,n_4,n_5,n_6) \times \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6) \\ =: I + II. \tag{1.65}$$

Before we proceed, we need to prove pointwise bounds on the multiplier M_4. In order to do this, let $(n_1,n_2,n_3,n_4) \in \Gamma_4$ be given. We dyadically localize the frequencies, i.e, we find dyadic integers N_j s.t. $|n_j| \sim N_j$. At this point it is important to order the frequencies N_j by their sizes. To do this we define $N_1^* := \max_{i=1,\dots,4} N_i$, N_2^* the maximum of the remaining ones and so on, as a result $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. We slightly abuse notation by writing $\theta(N_j^*)$ for $\theta(N_j^*, 0)$.

Lemma 1.20. *With notation as above, the following bound holds:*

$$M_4 = O\left(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\right). \tag{1.66}$$

Proof. By construction of the set Ω_{nr}, we note that:

$$|M_4| \lesssim \frac{|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2|}{|n_{12}||n_{14}|\beta_0}. \tag{1.67}$$

Let us assume that:

$$|n_1| \geq |n_2|, |n_3|, |n_4|, \text{ and } |n_{12}| \geq |n_{14}|, \tag{1.68}$$

the other cases are similar. We now have to consider three cases:

Case 1: $|n_1| \sim |n_{12}| \sim |n_{14}|$

In this Case, one has:

$$M_4 = O\Big(\frac{1}{\beta_0}\frac{(\theta(n_1))^2}{|n_1|^2}\Big) = O\Big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\Big).$$

Case 2: $|n_1| \sim |n_{12}| \gg |n_{14}|$

We use the *Mean Value Theorem*, and monotonicity properties of the function $\frac{(\theta(n))^2}{|n|}$ to deduce:

$$(\theta(n_1))^2 - (\theta(n_4))^2 = (\theta(n_1))^2 - (\theta(n_1 - n_{14}))^2 = O\big(|n_{14}|\frac{(\theta(n_1))^2}{|n_1|}\big). \tag{1.69}$$

Similarly

$$\begin{aligned}(\theta(n_2))^2 - (\theta(n_3))^2 &= (\theta(n_3 + n_{14}))^2 - (\theta(n_3))^2 \\ &= \left(|n_{14}| \sup_{N \leq |z| \lesssim |n_1|} \frac{(\theta(z))^2}{|z|}\right) = O\left(|n_{14}|\frac{(\theta(n_1))^2}{|n_1|}\right).\end{aligned} \tag{1.70}$$

Using (1.67), (1.69), (1.70), and the fact that $|n_{12}| \sim |n_1|$, it follows that:

$$M_4 = O\left(\frac{(\theta(n_1))^2}{|n_1|^2\beta_0}\right) = O\left(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\right).$$

Case 3: $|n_1| \gg |n_{12}|, |n_{14}|$

We write:

$$\begin{aligned}(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 &= (\theta(n_1))^2 - (\theta(n_1 - n_{12}))^2 \\ + (\theta(n_1 - n_{12} - n_{14}))^2 - (\theta(n_1 - n_{14}))^2.\end{aligned}$$

By using the *Double Mean-Value Theorem*, it follows that this expression is $O\left(\frac{(\theta(n_1))^2}{|n_1|^2}|n_{12}||n_{14}|\right)$. Consequently:

$$M_4 = O\left(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\right).$$

The Lemma now follows.

Let us choose:

$$\beta_0 \sim \frac{1}{N}. \tag{1.71}$$

The reason why we choose such a β_0 will become clear later. For details, see Remark 1.23. Hence Lemma 1.20 implies:

$$M_4 = O\left(\frac{N}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\right). \tag{1.72}$$

The bound from (1.72) allows us to deduce the equivalence of E^1 and E^2. We have the following bound:

Proposition 1.21. *For each fixed time t, one has:*

$$E^1(u(t)) \sim E^2(u(t)). \tag{1.73}$$

Here, the constant is independent of t and N, as long as N is sufficiently large.

This claim is proved by dyadically decomposing the factors u in frequency space and summing the appropriate components. We omit the details.

The iteration bound:

Let $\delta > 0, v$ be as in Proposition 1.17. For $t_0 \in \mathbb{R}$, we are interested in estimating:

$$E^2(u(t_0+\delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(u(t))dt = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(v(t))dt.$$

The iteration bound that we will show is:

Lemma 1.22. *For all $t_0 \in \mathbb{R}$, one has:*

$$\left|E^2(u(t_0+\delta)) - E^2(u(t_0))\right| \lesssim \frac{1}{N^{1-}}E^2(u(t_0)),$$

where N is the parameter we used to define the multiplier $\theta(n)$ in (1.49).

Arguing similarly as in [53, 54], Theorem 1.14 will follow from Lemma 1.22. We recall the proof for completeness.

Proof (Theorem 1.14 assuming Lemma 1.22).

The point is that we can iterate the following bound (obtained from Lemma 1.22):

$$E^2(u(t_0+\delta)) \leq (1+\frac{C}{N^{1-}})E^2(u(t_0))$$

$\sim N^{1-}$ times with a uniform time step, and the size of $E^2(t)$ will grow by at most a constant factor (and not as an exponential function in t). Hence we obtain that for $T \sim N^{1-}$, one has:

$$\|\mathcal{D}u(T)\|_{L^2} \lesssim \|\mathcal{D}\Phi\|_{L^2}.$$

By recalling (1.51), it follows that:

$$\|u(T)\|_{H^s} \lesssim N^s \|\Phi\|_{H^s}$$

and hence:

$$\|u(T)\|_{H^s} \lesssim T^{s+}\|\Phi\|_{H^s} \lesssim (1+T)^{s+}\|\Phi\|_{H^s}.$$

This proves Theorem 1.14 for times $t \geq 1$. The claim for times $t \in [0,1]$ follows by local well-posedness theory. The claim for negative times holds by time-reversibility.

We now have to prove Lemma 1.22.

Proof (Lemma 1.22).
Let us without loss of generality consider $t_0 = 0$. The general claim will follow by time translation, and the fact that all of the implied constants are uniform in time. Let v be the function constructed in Proposition 1.17, corresponding to $t_0 = 0$.

By (1.65), and with notation as in this equation, we need to estimate:

$$\int_0^\delta \Big(\sum_{\Gamma_4, |cos\angle(n_{12},n_{14})| \leq \beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) \widehat{v}(n_1)\widehat{\overline{v}}(n_2)\widehat{v}(n_3)\widehat{\overline{v}}(n_4)$$
$$+ \sum_{\Gamma_6} M_6(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{v}(n_1)\widehat{\overline{v}}(n_2)\widehat{v}(n_3)\widehat{\overline{v}}(n_4)\widehat{v}(n_5)\widehat{\overline{v}}(n_6)\Big) dt$$
$$= \int_0^\delta I dt + \int_0^\delta II dt =: A + B.$$

We now have to estimate A and B separately. Throughout our calculations, let us denote by $\chi = \chi(t) = \chi_{[0,\delta]}(t)$.

Estimate of A (Quadrilinear Terms)

By symmetry, we can consider without loss of generality the contribution when:

$$|n_1| \geq |n_2|, |n_3|, |n_4|, \text{and } |n_2| \geq |n_4|.$$

We note that when all $|n_j| \leq N$, one has:

$$(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = 0.$$

Hence, we need to consider the contribution in which one has:

$$|n_1| > N, |cos\angle(n_{12}, n_{14})| \leq \beta_0.$$

We dyadically localize the frequencies: $|n_j| \sim N_j$ for $j = 1, \ldots, 4$. We order the N_j to obtain $N_j^* \geq N_2^* \geq N_3^* \geq N_4^*$. Since $n_1 + n_2 + n_3 + n_4 = 0$, we know that

$$N_1^* \sim N_2^* \gtrsim N. \tag{1.74}$$

Let us note that $N_1 \sim N_2$. Namely, if it were the case that: $N_1 \gg N_2$, then, one would also have: $N_1 \gg N_4$, and the vectors n_{12} and n_{14} would form a very small angle. Hence, $\cos\angle(n_{12}, n_{14})$ would be close to 1, which would be a contradiction to the assumption that $|\cos\angle(n_{12}, n_{14})| \leq \beta_0$. Consequently

$$N_1 \sim N_2 \sim N_1^* \gtrsim N. \tag{1.75}$$

We denote the corresponding contribution to A by A_{N_1,N_2,N_3,N_4}. In other words

$$A_{N_1,N_2,N_3,N_4} :=$$
$$\int_0^\delta \sum_{\Gamma_4=0, |\cos\angle(n_{12},n_{14})| \leq \beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)$$
$$\times \widehat{v}_{N_1}(n_1)\widehat{\bar{v}}_{N_2}(n_2)\widehat{v}_{N_3}(n_3)\widehat{\bar{v}}_{N_4}(n_4)dt.$$

Arguing analogously as in the proof of Lemma 1.20, it follows that for the n_j that occur in the above sum, one has

$$\big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) = O\left(|n_{12}||n_{14}|\frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}\right). \tag{1.76}$$

By (1.75), it follows that $|n_3|, |n_4| \lesssim N_3^*$. Consequently

$$|n_{12}| = |n_{34}| \leq |n_3| + |n_4| \lesssim N_3^*.$$

One also knows that

$$|n_{14}| \leq |n_1| + |n_4| \lesssim N_1^*.$$

Substituting the last two inequalities into the multiplier bound (1.76), and using Parseval's identity in time, it follows that

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})| \leq \beta_0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} N_3^* N_1^* \frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}$$
$$\times |\widetilde{v}_{N_1}(n_1,\tau_1)||\widetilde{\bar{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||(\chi\bar{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j$$
$$\lesssim \frac{1}{N_1^*} \sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} |(\mathcal{D}v)\widetilde{\ }_{N_1}(n_1,\tau_1)||(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(n_2,\tau_2)|$$
$$\times |(\nabla v)\widetilde{\ }_{N_3}(n_3,\tau_3)||(\chi\bar{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j.$$

Let us define F_j for $j = 1, \ldots, 4$ by

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }_{N_1}|,\ \widetilde{F_2} := |(\mathcal{D}v)\widetilde{\ }_{N_2}|,\ \widetilde{F_3} := |(\nabla v)\widetilde{\ }_{N_3}|,\ \widetilde{F_4} := |(\chi v)\widetilde{\ }_{N_4}|.$$

Consequently, by Parseval's identity

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*} \int_{\mathbb{R}} \int_{\mathbb{T}^2} F_1 \overline{F_2} F_3 \overline{F_4} dxdt$$

By using an $L^4_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder inequality, we can continue with[10]

$$\lesssim \frac{1}{N_1^*} \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \|F_4\|_{L^4_{t,x}}$$

By using (1.36), and the fact that taking absolute values in the spacetime Fourier transforms doesn't change the $X^{s,b}$ norm, it follows that this is bounded by

$$\lesssim \frac{1}{N_1^*} \|\mathcal{D} v_{N_1}\|_{X^{0+,\frac{1}{2}+}} \|\mathcal{D} v_{N_2}\|_{X^{0+,\frac{1}{2}+}} \|v_{N_3}\|_{X^{1,\frac{1}{2}+}} \|v_{N_4}\|_{X^{0+,\frac{1}{2}+}}$$

By using frequency localization we can continue with

$$\lesssim \frac{1}{(N_1^*)^{1-}} \|\mathcal{D} v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^2_{X^{1,\frac{1}{2}+}} \lesssim \frac{1}{(N_1^*)^{1-}} E^1(\Phi).$$

In the last inequality, we used Proposition 1.17. By using the previous inequality, and by recalling (1.73), it follows that

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{(N_1^*)^{1-}} E^2(\Phi). \tag{1.77}$$

Using (1.77), summing in the N_j, and using (1.74) to deduce that

$$|A| \lesssim \frac{1}{N^{1-}} E^2(\Phi). \tag{1.78}$$

Estimate of *B* (Sextilinear Terms)

Let us consider just the first term in B coming from the summand $M_4(n_{123}, n_4, n_5, n_6)$ in the definition of M_6. The other terms are bounded analogously. In other words, we want to estimate

$$B^{(1)} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6) \times \widehat{(v\bar{v}v)}(n_1+n_2+n_3) \widehat{\bar{v}}(n_4) \widehat{v}(n_5) \widehat{\bar{v}}(n_6) dt.$$

We now dyadically localize n_{123}, n_4, n_5, n_6, i.e., we find N_j for $j = 1, \ldots, 4$ such that

[10] Strictly speaking, we are using an $L^4_{t,x}, L^4_{t,x}, L^{4+}_{t,x}, L^{4-}_{t,x}$ Hölder inequality, as well as estimates similar to (1.36) to estimate the $L^{4+}_{t,x}$, and $L^{4-}_{t,x}$ norm and appropriate time-localization properties of the $X^{s,b}$ spaces. We omit the details.

$$|n_{123}| \sim N_1, |n_4| \sim N_2, |n_5| \sim N_3, |n_6| \sim N_4.$$

Let us define

$$B^{(1)}_{N_1,N_2,N_3,N_4} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6) \\ \times \widehat{(v\overline{v}v)}_{N_1}(n_1+n_2+n_3)\widehat{\overline{v}}_{N_2}(n_4)\widehat{v}_{N_3}(n_5)\widehat{\overline{v}}_{N_4}(n_6)dt.$$

We now order the N_j to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. As before, we know the following localization bound

$$N_1^* \sim N_2^* \gtrsim N. \tag{1.79}$$

In order to obtain a bound on the wanted term, we have to consider two cases, depending on whether N_1 is among the two larger frequencies or not. An argument similar to the estimate of the quadrilinear terms gives

$$|B_{N_1,N_2,N_3,N_4}| \lesssim \frac{N}{(N_1^*)^{2-}} E^2(u_0). \tag{1.80}$$

We now use (1.80), sum in the N_j, and recall (1.79) to deduce that

$$|B| \lesssim \frac{1}{N^{1-}} E^2(u_0). \tag{1.81}$$

The Lemma now follows from (1.78) and (1.81).

Remark 1.23. The quantity β_0 was chosen as in (1.71) in order to get the same decay factor in the quantities A and B. We note that the quantity β_0 only occurred in the bound for B, whereas in the bound for A, we only used the fact that the terms corresponding to the largest two frequencies in the multiplier $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ appear with an opposite sign.

1.4.3 Energy transfer to high frequencies

In this section we show that a very weak growth of Sobolev norms may indeed occur. More precisely we can prove

Theorem 1.24. *[Colliander-Keel-Staffilani-Takaoka-Tao, [27]] Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the IVP (1.41) and $T > 0$ such that*

$$\|u_0\|_{H^s} \leq \sigma \quad \textit{and} \quad \|u(T)\|^2_{\dot{H}^s} \geq K.$$

We start by listing the elements of the proof. The first is a reduction to a resonant problem that we will refer to as the RFNLS system, see (1.83). Then in Subsection 1.4.4 we construct a special finite set Λ of frequencies and we reduce to a resonant, finite-dimensional Toy Model ODE system, see (1.84). We study this Toy Model dynamically and we show some sort of "Arnold diffusion" for it, see Theorem 1.25. Finally an the approximation Lemma 1.26 together with a scaling argument will finish the job.

We consider the gauge transformation

$$v(t,x) = e^{-i2Gt}u(t,x),$$

for $G \in \mathbb{R}$. If u solves the NLS above, then v solves the equation

$$(-i\partial_t + \Delta)v = (2G + v)|v|^2.$$

We make the ansatz

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(\langle n,x\rangle + |n|^2 t)}.$$

Now the dynamics is all recast trough $a_n(t)$:

$$i\partial_t a_n = 2Ga_n + \sum_{n_1 - n_2 + n_3 = n} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t},$$

where $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$. By choosing

$$G = -\|v(t)\|_{L^2}^2 = -\sum_k |a_k(t)|^2$$

which is constant from the conservation of the mass, one can rewrite the equation above as

$$i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t}, \tag{1.82}$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 \;/\; n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n\}.$$

From now on we will be referring to this system as the $FNLS$ system, with the obvious connection with the original NLS equation.

We define the set

$$\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) \,/\, \omega_4 = 0\}.$$

The geometric interpretation for this set is the following: If n_1, n_2, n_3 are in $\Gamma_{res}(n)$, then these four points represent the vertices of a rectangle in $\mathbb{Z}^2$. We finally define the Resonant Truncation $RFNLS$ to be the system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b}_{n_2} b_{n_3}. \tag{1.83}$$

We now would like to restrict the dynamics to a finite set of frequencies and this set would need several important properties. The first property we would require is closedness under resonancy. Thus we say a finite set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \implies n = n_1 - n_2 + n_3 \in \Lambda.$$

Restricting our attention to such Λ we introduce a Λ-finite dimensional resonant truncation of $RFNLS$ as

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Gamma_{res}(n)\cap\Lambda^3} b_{n_1}\overline{b}_{n_2}b_{n_3}. \tag{1.84}$$

We will refer to this systems as the $RFNLS_\Lambda$ system.

1.4.4 Λ: a very special set of frequencies

We can construct [27] a special set Λ of frequencies with the following properties

- **Generational set up:** $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$, N to be fixed later, where a nuclear family is a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the *parents*) live in generation Λ_j and n_2, n_4 (the *children*) live in generation Λ_{j+1}.
- **Existence and Uniqueness of Spouse and Children:** $\forall\ 1 \le j < M$ and $\forall\ n_1 \in \Lambda_j$ $\exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- **Existence and Uniqueness of Siblings and Parents:** $\forall\ 1 \le j < M$ and $\forall\ n_2 \in \Lambda_{j+1}$ $\exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- **Non Degeneracy:** The sibling of a frequency is never its spouse.
- **Faithfulness:** Besides nuclear families, Λ contains no other rectangles.
- **Integenerational Equality:** The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.
- **Multiplicative Structure:** If $N = N(\sigma, K)$ is large enough then Λ consists of $N \times 2^{N-1}$ disjoint frequencies n with $|n| > R = R(\sigma, K)$, the first frequency in Λ_1 is of size R and we call R the *Inner Radius* of Λ. Moreover for any $n \in \Lambda$, $|n| \le C(N)R$.
- **Wide Spreading:** Given $\sigma \ll 1$ and $K \gg 1$, if N is large enough then $\Lambda = \Lambda_1 \cup .... \cup \Lambda_N$ as above and

$$\sum_{n\in\Lambda_N} |n|^{2s} \ge \frac{K^2}{\sigma^2} \sum_{n\in\Lambda_1} |n|^{2s}.$$

- **Approximation:** If supp $(a_n(0)) \subset \Lambda$ then $FNLS$-evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $RFNLS_\Lambda$-ODE $a_n(0) \longmapsto b_n(t)$.

The Toy Model

The intergenerational equality hypothesis that the function $n \longmapsto b_n(0)$ is constant on each generation Λ_j) persists under $RFNLS_\Lambda$ (1.84):

$$\forall\ m, n \in \Lambda_j,\ b_n(t) = b_m(t).$$

Also $RFNLS_\Lambda$ may be reindexed by generation number j and the recast dynamics is the Toy Model is:

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)}, \tag{1.85}$$

with the boundary condition

$$b_0(t) = b_{N+1}(t) = 0. \tag{1.86}$$

One can simply show that the following are conserved quantities for the Toy Model

$$\begin{aligned} \text{Mass} &= \sum_j |b_j(t)|^2 = C_0 \\ \text{Momentum} &= \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} n = C_1, \end{aligned}$$

and if

$$\begin{aligned} \text{Kinetic Energy } &= \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} |n|^2 \\ \text{Potential Energy} &= \frac{1}{2}\sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2|b_{j+1}(t)|^2, \end{aligned}$$

then their sum, the Energy of the system, is conserved.

Using direct calculation[11], we will prove[12] that our Toy Model evolution $b_j(0) \longmapsto b_j(t)$ is such that:

$$\begin{aligned} (b_1(0), b_2(0), \ldots, b_N(0)) &\sim (1, 0, \ldots, 0) \\ (b_1(t_2), b_2(t_2), \ldots, b_N(t_2)) &\sim (0, 1, \ldots, 0) \\ &\cdot \\ (b_1(t_N), b_2(t_N), \ldots, b_N(t_N)) &\sim (0, 0, \ldots, 1) \end{aligned}$$

that is the bulk of conserved mass is transferred from Λ_1 to Λ_N and the weak transfer of energy from lower frequencies to higher follows from the *wide diaspora* property of Λ listed above.

[11] Maybe dynamical systems methods are useful here?
[12] See Theorem 1.25.

Global well-posedness for the Toy Model (1.85) is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^N \ / \ |x|^2 = 1\} \text{ and the flow map } W(t) : \Sigma \to \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of ODE. It is easy to see that for any $b \in \Sigma$

$$\partial_t |b_j|^2 = 4\text{Re}(i\bar{b_j}^2(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 \implies b_j(t) = 0, \quad \text{for all} \quad t \in [0, T].$$

If moreover we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_N) \in \Sigma \,/\, |b_j| = 1, \, b_k = 0, \, k \neq j\}$$

then

$$W(t)\mathbb{T}_j = \mathbb{T}_j \quad \text{for all } j = 1, ..., N$$

($\mathbb{T}_j$ is invariant). This suggests that if we want to move from a torus $\mathbb{T}_j$ to another we cannot start from data on the tori. Moreover, we need to show that we can manage to avoid hitting any one of them. This is in fact the content of the following instability-type theorem:

Theorem 1.25. *[Sliding Theorem] Let $N \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{N-2} within ϵ of $\mathbb{T}_{N-2}$ and a time τ such that*

$$W(\tau)x_3 = x_{N-2}.$$

What the theorem says is that $W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time t_0 which then gets moved so that it is concentrated near mode $j = N - 2$ at later time τ.

For the complete, and unfortunately lengthy proof of this theorem see [27]. Here we only give a motivation for it which should clarify the dynamics involved. Let us first observe that when $N = 2$ we can easily demonstrate that there is an orbit connecting $\mathbb{T}_1$ to $\mathbb{T}_2$. Indeed in this case we have the explicit "slider" solution

$$b_1(t) := \frac{e^{-it}\omega}{\sqrt{1 + e^{2\sqrt{3}t}}}; \quad b_2(t) := \frac{e^{-it}\omega^2}{\sqrt{1 + e^{-2\sqrt{3}t}}} \tag{1.87}$$

where $\omega := e^{2\pi i/3}$ is a cube root of unity.

This solution approaches $\mathbb{T}_1$ exponentially fast as $t \to -\infty$, and approaches $\mathbb{T}_2$ exponentially fast as $t \to +\infty$. One can translate this solution in the j parameter, and obtain solutions that "slide" from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$. Intuitively, the proof of the Sliding Theorem for higher N should then proceed by concatenating these slider solutions....This cannot work directly though because each solution requires an infinite amount of time to connect one circle to the next but it turns out that a suitably perturbed or "fuzzy" version of these slider solutions can in fact be glued together.

The Approximation Lemma and the Scaling Argument

Lemma 1.26. *[Approximation] Let $\Lambda \subset \mathbb{Z}^2$ introduced above. Let $B \gg 1$ and $\delta > 0$ small and fixed. Let $t \in [0,T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Lambda)$ solving $RFNLS_\Lambda$ such that*

$$\|b(t)\|_{l^1} \lesssim B^{-1}.$$

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of $FNLS$ (1.82) such that

$$a(0) = b(0), \quad \textit{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta},$$

for any $t \in [0,T]$.

The proof for this lemma is pretty standard. The main idea is to check that the "non resonant" part of the nonlinearity remains small enough, see [27] for details.

We are now ready to recast the main theorem with all the notations and reductions introduced so far:

Theorem 1.27. *For any $0 < \sigma \ll 1$ and $K \gg 1$ there exists a complex sequence (a_n) such that*

$$\left(\sum_{n\in\mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ of $FNLS$ and $T > 0$ such that

$$\left(\sum_{n\in\mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$

We need one last ingredient before we proceed to the proof of our main result: *the scaling argument*. In order to be able to use "instability" to move mass from lower frequencies to higher ones and start with a small data we need to introduce scaling. It is easy to check that if $b(t)$ solves $RFNLS_\Lambda$ (1.84) then the rescaled function

$$b^\lambda(t) = \lambda^{-1} b(\frac{t}{\lambda^2})$$

solves the same system with datum $b_0^\lambda = \lambda^{-1} b_0$.

We then pick the complex vector $b(0)$ that was found in the Sliding Theorem 1.25 above. For simplicity let us assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} b_j^\lambda(0) & \text{for any} \;\; n \in \Lambda_j \\ 0 & \text{otherwise} \; . \end{cases}$$

Proof Theorem 1.27

We start by estimating the size of $a(0)$. By definition

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\frac{1}{\lambda}\left(\sum_{j=1}^{M}|b_j(0)|^2\left(\sum_{n\in\Lambda_j}|n|^{2s}\right)\right)^{1/2}\sim\frac{1}{\lambda}Q_3^{1/2},$$

where we define

$$\sum_{n\in\Lambda_j}|n|^{2s}=Q_j,$$

and $a_n(0)$ is as above. At this point we use the proprieties of the set Λ to estimate $Q_3C(N)R^{2s}$, where R is the inner radius of Λ. We then conclude that

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\lambda^{-1}C(N)R^s\sim\sigma,$$

for a large enough R.

Now we want to estimating the size of $a(T)$. By using the perturbation lemma with $B=\lambda$ and $T=\lambda^2\tau$ we have

$$\|a(T)\|_{H^s}\geq\|b^\lambda(T)\|_{H^s}-\|a(T)-b^\lambda(T)\|_{H^s}=I_1-I_2.$$

We want $I_2\ll 1$ and $I_1>K$. For I_2 we use the Approximation Lemma

$$I_2\lesssim\lambda^{-1-\delta}\left(\sum_{n\in\Lambda}|n|^{2s}\right)^{1/2}.$$

As above,

$$I_2\lesssim\lambda^{-1-\delta}C(N)R^s.$$

At this point we need to pick λ and N so that

$$\|a(0)\|_{H^s}=\lambda^{-1}C(N)R^s\sim\sigma\quad\text{and}\quad I_2\lesssim\lambda^{-1-\delta}C(N)R^s\ll 1$$

and thanks to the presence of $\delta>0$ this can be achieved by taking λ and R large enough.

Finally we estimate I_1. It is important here that at time zero one starts with a fixed, non-zero datum, namely $\|a(0)\|_{H^s}=\|b^\lambda(0)\|_{H^s}\sim\sigma>0$. In fact we will show that

$$I_1^2=\|b^\lambda(T)\|_{H^s}^2\geq\frac{K^2}{\sigma^2}\|b^\lambda(0)\|_{H^s}^2\sim K^2.$$

For $T=\lambda^2t$ define

$$R=\frac{\sum_{n\in\Lambda}|b_n^\lambda(\lambda^2t)|^2|n|^{2s}}{\sum_{n\in\Lambda}|b_n^\lambda(0)|^2|n|^{2s}},$$

then it remains to show that $R \gtrsim K^2/\sigma^2$. Recall the notation

$$\Lambda = \Lambda_1 \cup ..... \cup \Lambda_N \quad \text{and} \quad \sum_{n\in\Lambda_j} |n|^{2s} = Q_j.$$

Using the fact that by the Sliding Theorem 1.25 one obtains $b_j(T) = 1-\epsilon$ if $j = N-2$ and $b_j(T) = \epsilon$ if $j \neq N-2$, it follows that

$$\begin{aligned} R &= \frac{\sum_{i=1}^N \sum_{n\in\Lambda_i} |b_i^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^N \sum_{n\in\Lambda_i} |b_i^\lambda(0)|^2 |n|^{2s}} \\ &\geq \frac{Q_{N-2}(1-\epsilon)}{(1-\epsilon)Q_3 + \epsilon Q_1 + ... + \epsilon Q_N} \sim \frac{Q_{N-2}(1-\epsilon)}{Q_{N-2}\left[(1-\epsilon)\frac{Q_3}{Q_{N-2}} + ... + \epsilon\right]} \\ &\gtrsim \frac{(1-\epsilon)}{(1-\epsilon)\frac{Q_3}{Q_{N-2}}} = \frac{Q_{N-2}}{Q_3} \end{aligned}$$

and the conclusion follows from the "Wide Spreading" property of Λ_j:

$$Q_{N-2} = \sum_{n\in\Lambda_{N-2}} |n|^{2s} \gtrsim \frac{K^2}{\sigma^2} \sum_{n\in\Lambda_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3.$$

On the Construction of the set Λ

It is not obvious to see how one could manage to construct a set Λ with the properties listed above. Where does the set Λ come from?

Here we do not construct Λ, but we construct Σ, a set that has a lot of the properties of Λ, but "lives" in a different space. For a full construction of the set Λ one should consult [27].

We define the *standard unit square* $S \subset \mathbb{C}$ to be the four-element set of complex numbers

$$S = \{0, 1, 1+i, i\}.$$

We split $S = S_1 \cup S_2$, where $S_1 := \{1, i\}$ and $S_2 := \{0, 1+i\}$. The combinatorial model Σ is a subset of a large power of the set S. More precisely, for any $1 \leq j \leq N$, we define $\Sigma_j \subset \mathbb{C}^{N-1}$ to be the set of all $N-1$-tuples $(z_1, \ldots, z_{N-1})$ such that $z_1, \ldots, z_{j-1} \in S_2$ and $z_j, \ldots, z_{N-1} \in S_1$. In other words,

$$\Sigma_j := S_2^{j-1} \times S_1^{N-j}.$$

Note that each Σ_j consists of 2^{N-1} elements, and they are all disjoint. We then set $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$; this set consists of $N2^{N-1}$ elements. We refer to Σ_j as the j^{th} *generation* of Σ.

For each $1 \leq j < N$, we define a *combinatorial nuclear family connecting generations* Σ_j, Σ_{j+1} to be any four-element set $F \subset \Sigma_j \cup \Sigma_{j+1}$ of the form

$$F := \{(z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N) : w \in S\}$$

where $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_N \in S_1$ are fixed. In other words, we have

$$F = \{F_0, F_1, F_{1+i}, F_i\} = \{(z_1, \ldots, z_{j-1})\} \times S \times \{(z_{j+1}, \ldots, z_N)\}$$

where $F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N)$.

It is clear that F is a four-element set consisting of two elements F_1, F_i of Σ_j (which we call the *parents* in F) and two elements F_0, F_{1+i} of Σ_{j+1} (which we call the *children* in F). Also for each j there are 2^{N-2} combinatorial nuclear families connecting the generations Σ_j and Σ_{j+1}.

One easily verifies the following properties:

- **Existence and uniqueness of spouse and children**: For any $1 \leq j < N$ and any $x \in \Sigma_j$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that x is a parent of this family (i.e. $x = F_1$ or $x = F_i$). In particular each $x \in \Sigma_j$ has a unique spouse (in Σ_j) and two unique children (in Σ_{j+1}).
- **Existence and uniqueness of sibling and parents**: For any $1 \leq j < N$ and any $y \in \Sigma_{j+1}$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that y is a child of the family (i.e. $y = F_0$ or $y = F_{1+i}$). In particular each $y \in \Sigma_{j+1}$ has a unique sibling (in Σ_{j+1}) and two unique parents (in Σ_j).
- **Nondegeneracy**: The sibling of an element $x \in \Sigma_j$ is never equal to its spouse.

We conclude this section with an example

Example: If $N = 7$, the point $x = (0, 1+i, 0, i, i, 1)$ lies in the fourth generation Σ_4. Its spouse is $(0, 1+i, 0, 1, i, 1)$ (also in Σ_4) and its two children are $(0, 1+i, 0, 0, i, 1)$ and $(0, 1+i, 0, 1+i, i, 1)$ (both in Σ_5). These four points form a combinatorial nuclear family connecting the generations Σ_4 and Σ_5. The sibling of x is $(0, 1+i, 1+i, i, i, 1)$ (also in Σ_4, but distinct from the spouse) and its two parents are $(0, 1+i, 1, i, i, 1)$ and $(0, 1+i, i, i, i, 1)$ (both in Σ_3). These four points form a combinatorial nuclear family connecting the generations Σ_3 and Σ_4. Elements of Σ_1 do not have siblings or parents, and elements of Σ_7 do not have spouses or children.

1.5 Periodic Schrödinger equations and Gibbs measures

We already recalled in Subsection 1.2.2 that certain periodic NLS equations can be viewed as infinite dimensional Hamiltonian systems. We now recall few basic facts which are true for finite dimensional Hamiltonian systems.

1.5.1 The finite dimension case

Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i} \tag{1.88}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \; = \; \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}(-\frac{\partial H}{\partial q_i}) = 0.$$

By defining $y := (q_1, \dots, q_k, p_1, \dots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We now recall the following theorem giving a sufficient condition under which a flow map preserves the volume:

Theorem 1.28 (Liouville's Theorem). *Let a vector field* $f : \mathbb{R}^d \to \mathbb{R}^d$ *be divergence free. If the flow map* Φ_t *satisfies:*

$$\frac{d}{dt}\Phi_t(y) \; = \; f(\Phi_t(y)),$$

then for all t *it is a volume preserving map.*

In particular if f is associated to a Hamiltonian system then automatically $div\, f = 0$. Indeed

$$div\, f = \frac{\partial}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial}{\partial q_2}\frac{\partial H}{\partial p_2} + \dots \frac{\partial}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial}{\partial p_2}\frac{\partial H}{\partial q_2} - \dots \frac{\partial}{\partial p_k}\frac{\partial H}{\partial q_k} = 0$$

by equality of mixed partial derivatives. As a consequence, the Lebesgue measure on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow of (1.88).

There is another measures that remain invariant[13] under the Hamiltonian flow: the Gibbs measures

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i \, dq_i$$

with $\beta > 0$.

From conservation of Hamiltonian H the function $e^{-\beta H(p,q)}$ remains constant, while, thanks to the Liouville Theorem, the volume $\prod_{i=1}^{d} dp_i \, dq_i$ remains invariant as well.

---

[13] A measure μ remains invariant under a flow Φ_t if for any A, subset of the support of μ one has

$$\mu(\Phi_t(A)) = \mu(A).$$

1.5.2 The periodic, one dimensional nonlinear Schrödinger equation

In the context of semilinear NLS, as mentioned in Subsection 1.2.2, one can think of u, solution to $iu_t + u_{xx} \pm |u|^{p-1}u = 0$ on $\mathbb{T}$, as the infinite dimension vector given by its Fourier coefficients:

$$\hat{u}(n) = a_n + ib_n, \quad n \in \mathbb{Z}$$

and with respect to the Hamiltonian

$$H(u) = \frac{1}{2}\int |u_x|^2\,dx \pm \frac{1}{p}\int |u|^p\,dx$$

one can think of the equation as an infinite dimension Hamiltonian system.

Lebowitz, Rose and Speer [43] considered the Gibbs measure *formally* given by

$$'d\mu = Z^{-1}\exp\left(-\beta H(u)\right)\prod_{x\in\mathbb{T}} du(x)'$$

and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$ but not for $s = \frac{1}{2}$, see Remark 1.32.

Remark 1.29. In the focusing case the result only holds for $p \leq 5$ with the L^2-cutoff $\chi_{\|u\|_{L^2}\leq B}$ for any $B > 0$ if $p < 5$ and with small B for $p = 5$, see [5] (recall the L^2 norm is conserved for these equations.)

Bourgain [5] proved the invariance of this measure and almost surely global well-posedness of the associated initial value problem[14]. More precisely, in the focusing case, for example, he proved:

Theorem 1.30. *Consider the focusing NLS initial value problem*

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^4 u \\ u(0,x) = u_0(x), \quad \textit{where } x \in \mathbb{T}. \end{cases} \tag{1.89}$$

Then the measure μ introduced above is well defined in H^s, $0 < s < 1/2$ for B small and almost surely with respect to it the problem is globally well-posed. Moreover the measure μ is invariant under the flow given by (1.89).

Remark 1.31. In the defocusing case, the same theorem is true without the L^2 ball restriction.

Two elements of the theorem above are particularly relevant: the Global Well-Posedness and the Invariance of the Measure.

In the past few years two methods have been developed and applied to study the global in time existence of dispersive equations at regularities which

[14] Note that this fact is true both in the focusing and defocusing case.

are right below or in between those corresponding to conserved quantities: The high-low method by Bourgain [8] and the I-method (or method of *almost conservation laws*) by Colliander, Keel, Staffilani, Takaoka and Tao [21, 22, 24]. When these two methods fail, Bourgain's approach for periodic dispersive equations (NLS, KdV, mKdV, Zakharov system) has been through the introduction and use of the Gibbs measure derived from the PDE viewed as an infinite dimension Hamiltonian system. Why is this last method effective? There are two fundamental reasons. The first is that failure to show global existence by Bourgain's high-low method or the I-method might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure is that it does not see that exceptional set. the second is that the invariance of the Gibbs measure, just like the usual conserved quantities, can be used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely.

The difficulty in this approach lies in the actual construction of the associated Gibbs measure and in showing its invariance under the flow. Nevertheless this approach has recently successfully been used for example by T. Oh for the periodic KdV-type and Schrödinger-Benjamin-Ono on $\mathbb{T}$ coupled systems [45, 46, 47], by Burq and Tzevtkov for subcubic and subquartic radial NLW on 3d ball [14, 15] and by Thomann and Tzevtkov for DNLS (only formal construction of the measure) [62].

1.5.3 Gauss measure and Gibbs measures

Let's take the IVP in Theorem 1.30 above. Note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is conserved, but one usually sees the Gibbs measure μ written as

$$d\mu = Z^{-1}\chi_{\|u\|_{L^2}\le B} \exp\left(\frac{1}{6}\int |u|^6\,dx\right) \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right) \prod_{x\in\mathbb{T}} du(x)$$

where

$$d\rho = \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right) \prod_{x\in\mathbb{T}} du(x)$$

is the Gauss measure that is well understood in $H^s, s < 1/2$ and

$$\frac{d\mu}{d\rho} = \chi_{\|u\|_{L^2}\le B} \exp\left(\frac{1}{6}\int |u|^6\,dx\right),$$

corresponding to the nonlinear term of the Hamiltonian, is understood as the Radon-Nikodym derivative of μ with respect to ρ.

Our Gauss measure ρ is defined as weak limit of the finite dimensional Gauss measures

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\le N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\le N} da_n db_n \,.$$

Remark 1.32. We recall here that the measure ρ_N above can be regarded as the induced probability measure on $\mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{|n|\le N} \qquad \text{and} \qquad \widehat{v}_n = \frac{g_n}{\sqrt{1+|n|^2}},$$

where $\{g_n(\omega)\}_{|n|\le N}$ are independent standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$.

In a similar manner, we can view ρ as the induced probability measure under the map

$$\omega \mapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{n\in\mathbb{Z}} .$$

But what is its support? Consider the operator $\mathcal{J}_s = (1-\Delta)^{s-1}$ then

$$\sum_n (1+|n|^2)\,|\widehat{v}_n|^2 \;=\; \langle v, v\rangle_{H^1} \;=\; \langle \mathcal{J}_s^{-1} v, v\rangle_{H^s} \,.$$

The operator $\mathcal{J}_s : H^s \to H^s$ has the set of eigenvalues $\{(1+|n|^2)^{(s-1)}\}_{n\in\mathbb{Z}}$ and the corresponding eigenvectors $\{(1+|n|^2)^{-s/2}e^{inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of H^s if $s<1$. For ρ to be *countably additive* we need $\mathcal{J}_s$ to be of *trace class* which is true if and only if $s<\frac{1}{2}$. Then ρ is a countably additive measure on H^s for any $s<1/2$ (but **not** for $s \ge 1/2$!) See [42, 31].

1.5.4 Bourgain's Method

Above, we stated Bourgain's theorem for the quintic focusing periodic NLS. Here we give an outline of Bourgain's idea in a general framework, and discuss how to prove almost surely global well-posedness and the invariance of a measure starting with a local well-posedness result.

Consider a dispersive nonlinear Hamiltonian PDE with a k-linear nonlinearity, possibly with derivative:

$$\begin{cases} u_t = \mathcal{L}u + \mathcal{N}(u) \\ u|_{t=0} = u_0 \end{cases} \tag{1.90}$$

where $\mathcal{L}$ is a (spatial) differential operator like $i\partial_{xx}$, ∂_{xxx}, etc. Let $H(u)$ denote the Hamiltonian of (1.90). Then (1.90) can also be written as

$$u_t = J\,\frac{dH}{du} \quad \text{if } u \text{ is real-valued,} \quad u_t = J\,\frac{\partial H}{\partial \overline{u}} \quad \text{if } u \text{ is complex-valued,}$$

for an appropriate operator J. Let μ denote a measure on the distributions on $\mathbb{T}$, whose invariance we would like to establish. We assume that μ is a weighted Gaussian measure (formally) given by

$$' d\mu = Z^{-1} e^{-F(u)} \prod_{x\in\mathbb{T}} du(x)',$$

where $F(u)$ is conserved[15] under the flow of (1.90) and the leading term of $F(u)$ is quadratic and nonnegative. Now, suppose that there is a good local well-posedness theory, that is, there exists a Banach space $\mathcal{B}$ of distributions on $\mathbb{T}$ and a space $X_\delta \subset C([-\delta,\delta];\mathcal{B})$ of space-time distributions in which to prove local well-posedness by a fixed point argument with a time of existence δ depending on $\|u_0\|_B$, say $\delta \sim \|u_0\|_B^{-\alpha}$ for some $\alpha > 0$. In addition, suppose that the Dirichlet projections P_N – the projection onto the spatial frequencies $\leq N$ – act boundedly on these spaces, uniformly in N. Then for $\|u_0\|_B \leq K$ the finite dimensional approximation to (1.90)

$$\begin{cases} u_t^N = \mathcal{L}u^N + P_N\big(\mathcal{N}(u^N)\big) \\ u^N|_{t=0} = u_0^N := P_N u_0(x) = \sum_{|n|\leq N} \widehat{u}_0(n) e^{inx}. \end{cases} \tag{1.91}$$

is locally well-posed on $[-\delta,\delta]$ with $\delta \sim K^{-\alpha}$, independent of N.

We need two more important assumptions on (1.91): that (1.91) is Hamiltonian with $H(u^N)$ i.e.

$$u_t^N = J\,\frac{dH(u^N)}{d\overline{u}^N} \tag{1.92}$$

and that

$$\frac{d}{dt}F(u^N(t)) = 0, \tag{1.93}$$

that is $F(u^N)$ is still conserved under the flow of (1.91).

Note that the first holds for example when J commutes with the projection P_N, (e.g. $J = i$ or ∂_x.). In general however the two assumptions above are not guaranteed and may not necessarily hold. See Section 1.6.

From this point on the argument goes through the following steps:

- By Liouville's theorem and (1.92) the Lebesgue measure

  $$\prod_{|n|\leq N} da_n db_n,$$

  where $\widehat{u^N}(n) = a_n + ib_n$, is invariant under the flow of (1.91).
- Using (1.93) - the conservation of $F(u^N)$- the finite dimensional version μ_N of μ:

  $$d\mu_N = Z_N^{-1} e^{-F(u^N)} \prod_{|n|\leq N} da_n db_n$$

  is also invariant under the flow of (1.91).

The next ingredient we need is:

[15] $F(u)$ could be the Hamiltonian, but not necessarily!

Lemma 1.33 (Fernique-type tail estimate). *For K sufficiently large, we have*

$$\mu_N\big(\{\|u_0^N\|_{\mathcal{B}} > K\}\big) < Ce^{-CK^2},$$

where all constants are independent of N.

Proof. Here we only give the idea of the proof in the case of the quintic NLS equation introduced above, for a more complete argument see [5]. We in fact skip the heart of the matter which is to show that if B is small enough then

$$\exp\left\|\sum_{|n|\le N}\frac{g_n(\omega)}{(1+n^2)^{1/2}}e^{i2\pi xn}\right\|_{L^6}^6 \chi_{\{\sum_{|n|\le N}\frac{|g_n(\omega)|^2}{1+n^2}<B\}} \in L^1(d\omega)$$

and the bound is uniform in N. Now define

$$\Omega_{N,K} := \left\{(a_n)\ /\ \left\|\sum_{|n|\le N} a_n e^{inx}\right\|_{H^s} > K \quad \text{and } L^2 \text{ restriction}\right\}$$

then

$$\begin{aligned}\mu_n(\Omega_{N,K}) &= \int_{\Omega_{NK}} \exp\left\|\sum_{|n|\le N} a_n e^{i2\pi xn}\right\|_{L^6}^6 d\rho_N \\ &\le C\mathbb{P}_{\rho_N}\left[\{\omega\ /\ \left\|\sum_{|n|\le N} a_n e^{inx}\right\|_{H^s} > K\}\right]^{1/2} \le Ce^{-CK^2}\end{aligned}$$

since ρ_N is a Gaussian measure.

The lemma we just presented and the invariance of μ_N imply the following estimate controlling the growth of solution u^N to (1.91) [5].

Proposition 1.34. *Given $T < \infty, \varepsilon > 0$, there exists $\Omega_N \subset \mathcal{B}$ such that $\mu_N(\Omega_N^c) < \varepsilon$ and for $u_0^N \in \Omega_N$, (1.91) is well-posed on $[-T,T]$ with the growth estimate:*

$$\|u^N(t)\|_{\mathcal{B}} \lesssim \left(\log\frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for } |t|\le T.$$

Proof. Let $\Phi_N(t) =$ be the flow map of (FDA), and define

$$\Omega_N = \cap_{j=-[T/\delta]}^{[T/\delta]}\Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}}\le K\}).$$

By invariance of μ_N,

$$\mu(\Omega_N^c) = \sum_{j=-[T/\delta]}^{[T/\delta]} \mu_N\left(\Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}} > K\}\right) = 2[T/\delta]\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\})$$

This implies $\mu(\Omega_N^c) \lesssim \frac{T}{\delta}\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\}) \sim TK^\theta e^{-cK^2}$, and by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}$, we have $\mu(\Omega_N^c) < \varepsilon$. By construction, $\|u^N(j\delta)\|_{\mathcal{B}} \leq K$ for $j = 0, \cdots, \pm[T/\delta]$ and by local theory,

$$\|u^N(t)\|_{\mathcal{B}} \leq 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} \text{ for } |t| \leq T.$$

One then needs to prove that μ_N converges weakly to μ. This is standard and one can go back to the work of Zhidkov [66] for example. One defines

$$E_N = [e^{2\pi inx} \ / \ |n| \leq N],$$

and then shows that if $U \subset H^s$, $s < 1/2$ is open the two limits:

$$\rho(U) := \lim_{N\to\infty} \rho_N(U \cap E_N)$$
$$\mu(U) := \lim_{N\to\infty} \mu_N(U \cap E_N)$$

are defined. Going back to (1.90), essentially as a corollary of Proposition 1.34 one can then prove:

Corollary 1.35. *(a) Given $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \mathcal{B}$ with $\mu(\Omega_\varepsilon^c) < \varepsilon$ such that for $u_0 \in \Omega_\varepsilon$, (PDE) is globally well-posed with the growth estimate:*

$$\|u(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{1+|t|}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for all } t \in \mathbb{R}.$$

(b) $\|u-u^N\|_{C([-T,T];\mathcal{B}')} \to 0$ as $N \to \infty$ uniformly for $u_0 \in \Omega_\varepsilon$, where $\mathcal{B}' \supset \mathcal{B}$.

One can prove (a) and (b) by estimating the difference $u - u^N$ using the local well-posedness theory and a standard approximation lemma, and then applying Proposition 1.34 to u^N. Finally if $\widetilde{\Omega} := \bigcup_{\varepsilon>0} \Omega_\varepsilon$, using (a) one obtains that $\mu(\widetilde{\Omega}) = 1$ and (1.90) is almost surely globally well-posed. At the same time one also obtains the invariance of μ.

1.6 The periodic, one dimensional derivative Schrödinger equation

It is now time to introduce another infinite dimensional system: the derivative NLS equation (DNLS)

$$\begin{cases} u_t - i\, u_{xx} = \lambda(|u|^2u)_x, \\ u\big|_{t=0} = u_0 \end{cases} \tag{1.94}$$

where either $(x,t) \in \mathbb{R} \times (-T,T)$ or $(x,t) \in \mathbb{T} \times (-T,T)$ and λ is real. Below we will take $\lambda = 1$ for convenience and note that DNLS is a Hamiltonian PDE with conservation of *mass* and *'energy'*. In fact, it is completely integrable [35]. The first three conserved quantities are:

$$\text{Mass:} \quad m(u) = \frac{1}{2\pi} \int_{\mathbb{T}} |u(x,t)|^2 \, dx$$

$$\text{Energy:} \quad E(u) = \int_{\mathbb{T}} |u_x|^2 \, dx + \frac{3}{2} \text{Im} \int_{\mathbb{T}} u^2 \overline{u u_x} \, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6 \, dx$$

$$\text{Hamiltonian:} \quad H(u) = \text{Im} \int_{\mathbb{T}} u \overline{u}_x \, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^4 \, dx$$

We would like now to explore the possibility of extending Bourgain's approach to the 1D periodic DNLS (1.94). Our goal is to construct an associated invariant weighted Wiener measure[16] and establish global well-posedness for data living in its support. Unfortunately the presence of the derivative term in (1.94) forces us to introduce a gauge transformation. It turns out that for the gauged DNLS the plan can be carried out [48]. The second goal is to show that the un-gauged invariant Wiener measure associated to the periodic derivative NLS obtained above is absolutely continuous with respect to the weighted Wiener measure constructed by Thomann and Tzvetkov [62]. This will be consequence of a more general result on absolute continuity of Gaussian measures under certain gauge transformations [51].

Let's now learn more about the DNLS equation introduced above. The equation is scale invariant for data in L^2: if $u(x,t)$ is a solution then $u_a(x,t) = a^{\frac{1}{2}} u(ax, a^2 t)$ is also a solution. Thus *a priori* one expects some form of existence and uniqueness for data in $H^\sigma, \sigma \geq 0$. Many results are known for the Cauchy problem with smooth data, including data in H^1 (Tsutsumi and Fukada [63]; Hayashi [34], Hayashi and Ozawa [35, 36] and Ozawa [50]). In looking for solutions to DNLS we face a derivative loss arising from the nonlinear term and hence for low regularity data the key is to somehow make up for this loss.

In the non-periodic case Takaoka [59, 60] proved sharp local well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$ via a gauge transformation, already introduced by Hayashi and Ozawa [35, 36], and sharp multilinear estimates for the gauged equivalent equation in the Fourier restriction norm spaces $X^{s,b}$. Later Colliander, Keel, Staffilani, Takaoka and Tao [21, 22] established global well-posedness in $H^\sigma(\mathbb{R})$, $\sigma > \frac{1}{2}$ of small[17] L^2 norm using the so-called I-method on the gauge equivalent equation. Miao, Wu and Xu [44] recently extended global well-posedness to $H^\sigma(\mathbb{R})$, $\sigma \geq \frac{1}{2}$. We also know that the Cauchy initial value problem is ill-posed for data in $H^\sigma(\mathbb{R})$ and $\sigma < \frac{1}{2}$; i.e. data map fails to be C^3 or uniformly C^0 (see Takaoka [59, 60], Biagioni and Linares [1]).

[16] Here we will be talking about Wiener measures instead of Gibbs measures since their supports will not be on Hilbert spaces but on Banach spaces. See for example Lemma 1.38.

[17] Here small in L^2 means $\lesssim \sqrt{\frac{2\pi}{\lambda}} \Rightarrow$ 'energy' to be positive via Gagliardo-Nirenberg inequality.

In the periodic case, S. Herr [37] showed that the Cauchy problem associated to DNLS (1.94) is locally well-posed for initial data $u(0) \in H^\sigma(\mathbb{T})$, if $\sigma \geq \frac{1}{2}$. The proof is based on an adaptation of the gauge transformation above mentioned to the periodic setting, see (1.96), and sharp multilinear estimates for the gauged equivalent equation in periodic Fourier restriction norm spaces $X^{s,b}$. By use of conservation laws, the problem is also shown to be globally well-posed for $\sigma \geq 1$ and data which are small in L^2-as in [21]. More recently Y. Y. Su Win [64] applied the I-method to prove GWP in $H^\sigma(\mathbb{T})$ for $\sigma > \frac{1}{2}$. Also, in the periodic case the problem is believed to be ill-posed in $H^\sigma(\mathbb{T})$ for $\sigma < \frac{1}{2}$ in the sense that fixed point theorem cannot be used with Sobolev spaces.

1.6.1 The periodic, one dimensional gauged derivative Schrödinger equation

Why do we need to gauge? Because the nonlinearity

$$(|u|^2 u)_x \;=\; u^2\,\overline{u}_x \;+\; 2\,|u|^2\,u_x \tag{1.95}$$

is hard to control due to the presence of the derivative. As will become clear below, both terms (1.95) aren't bad, only $|u|^2\,u_x$ and we remove it via an appropriate gauge transformation. In the periodic case the appropriate gauge transformation defined for functions $f \in L^2(\mathbb{T})$ is

$$G(f)(x) := \exp(-iJ(f))\,f(x) \tag{1.96}$$

where

$$J(f)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})} \right) dy\,d\theta$$

is the unique 2π-periodic mean zero primitive of the map

$$x \longrightarrow |f(x)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}.$$

Then, for $u \in C([-T,T];L^2(\mathbb{T}))$ the adapted periodic gauge is defined as

$$\mathcal{G}(u)(t,x) := G(u(t))(x - 2\,t\,m(u)),$$

where, as we will see later in (1.98), $m(u)$ represents the mass. We have that

$$\mathcal{G} : C([-T,T];H^\sigma(\mathbb{T})) \to C([-T,T];H^\sigma(\mathbb{T}))$$

is a homeomorphism for any $\sigma \geq 0$. Moreover, $\mathcal{G}$ is locally bi-Lipschitz on subsets of functions in $C([-T,T];H^\sigma(\mathbb{T}))$ with prescribed L^2-norm. The same is true if we replace $H^\sigma(\mathbb{T})$ by $\mathcal{F}L^{s,r}$, the Fourier-Lebesgue spaces defined in (1.101) below.

If u is a solution to (1.94) and $v := \mathcal{G}(u)$ we have that v solves what we refer to as the GDNLS equation:

$$v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v \tag{1.97}$$

with initial data $v(0) = \mathcal{G}(u(0))$ and where

$$m(u) = m(v) := \frac{1}{2\pi}\int_{\mathbb{T}} |v|^2(x,t)dx = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2(x)dx \tag{1.98}$$

and

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \operatorname{Im}(v\overline{v}_x)\, dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4 dx - m(v)^2.$$

Note both $m(v)$ and $\psi(v)(t)$ are real. It turned out that local well-posedness for the GDNLS in H^σ implies local existence and uniqueness for the DNLS (1.94) in H^σ; but we don't necessarily have all the auxiliary estimates coming from the local well-posedness result for the GDNLS (1.97).

What's the energy for GDNLS? For v the solution to the periodic GDNLS define

$$\mathcal{E}(v) := \int_{\mathbb{T}} |v_x|^2\, dx - \frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} v^2\overline{v\, v_x}\, dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v(t)|^2\, dx\right)\left(\int_{\mathbb{T}} |v(t)|^4\, dx\right),$$

$$\mathcal{H}(v) := \operatorname{Im}\int_{\mathbb{T}} v\overline{v}_x - \frac{1}{2}\int_{\mathbb{T}} |v|^4\, dx + 2\pi m(v)^2$$

and

$$\tilde{\mathcal{E}}(v) := \mathcal{E}(v) + 2m(v)\mathcal{H}(v) - 2\pi\, m(v)^3 \tag{1.99}$$

We prove:

$$\frac{d\tilde{\mathcal{E}}(v)}{dt} = 0. \tag{1.100}$$

In fact one can show that $E(u) = \tilde{\mathcal{E}}(v)$. We refer to $\tilde{\mathcal{E}}(v)$ from now on as the *energy* of GDNLS (1.97). Let us recall the local well-posedness result of Grünrock and Herr [33] which we will be using below. They showed that the Cauchy problem associated to DNLS is locally well-posed for initial data $u_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ and $2 \le r < 4$, $s \ge \frac{1}{2}$, where

$$\|u_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \|\, \langle\, n\, \rangle^s\, \hat{u}_0\, \|_{\ell^r_n(\mathbb{Z})} \qquad r \ge 2. \tag{1.101}$$

These spaces scale like the Sobolev spaces $H^\sigma(\mathbb{T})$, where $\sigma = s + 1/r - 1/2$. For example for $s = 2/3-$ and $r = 3$ one has that $\sigma < \frac{1}{2}$. The proof of this local well-posedness result is based on new and sharp multilinear estimates in an appropriate variant of Fourier restriction norm spaces $X^{s,b}_{r,q}$ introduced by Grünrock-Herr [33]:

$$\|u\|_{X^{s,b}_{r,q}} := \|\langle n\rangle^s \,\langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^r_n L^q_\tau}, \tag{1.102}$$

where first one takes the L^q_τ norm and then the ℓ^r_n one. For $\delta > 0$ fixed, the restriction space $X^{s,b}_{r,q}(\delta)$ is defined as usual through the norm

$$\|v\|_{X^{s,b}_{r,q}(\delta)} := \inf\{\|u\|_{X^{s,b}_{r,q}} \ : \ u \in X^{s,b}_{r,q} \ \text{ and } \ v = u\big|_{[-\delta,\delta]}\}. \tag{1.103}$$

For $q = 2$ we simply write $X^{s,b}_{r,2} = X^{s,b}_r$. Note $X^{s,b}_{2,2} = X^{s,b}$. Later we will also use the space

$$Z^s_r(\delta) := X^{s,\frac{1}{2}}_{r,2}(\delta) \cap X^{s,0}_{r,1}(\delta) \tag{1.104}$$

and we note that

$$Z^s_r(\delta) \subset C([-\delta,\delta], \mathcal{F}L^{s,r}).$$

Our first goal is now to establish the a.s GWP for the periodic DNLS in a Fourier Lebesgue space $\mathcal{F}L^{s,r}$ scaling below $H^{\frac{1}{2}}(\mathbb{T})$ and the invariance of the associate Gibbs measure μ. Here we recall again that an invariant measure μ means that if $\Phi(t)$ is the flow map associated to the nonlinear equation, then for reasonable F

$$\int F(\Phi(t)(\phi))\, \mu(d\phi) = \int F(\phi)\mu(d\phi).$$

As anticipated the method of proof will be to construct μ so that local well-posedness of periodic DNLS in some space B containing the support of μ holds; then show almost surely global well-posedness as well as the invariance of μ via both Bourgain's [5] and Zhidkov's [66] arguments.

Finite dimensional approximation of GDNLS (1.97)

We denote FGDNLS our finite dimensional approximation of (1.97):

$$v^N_t = iv^N_{xx} - P_N((v^N)^2\overline{v^N_x}) + \frac{i}{2}\, P_N(|v^N|^4 v^N) - i\psi(v^N)v^N - im(v^N)P_N(|v^N|^2 v^N) \tag{1.105}$$

with initial data $v^N_0 = P_N\, v_0$. Note $m(v^N)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} |v^N(x,t)|^2 dx$ is also conserved under the flow of (1.105).

Lemma 1.36 (Local well-posedness). *Let $2 < r < 4$ and $s \geq \frac{1}{2}$. Then for every*

$$v^N_0 \in B_R := \{v^N_0 \in \mathcal{F}L^{s,r}(\mathbb{T}) / \|v^N_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < R\}$$

and $\delta \lesssim R^{-\gamma}$, for some $\gamma > 0$, there exists a unique solution

$$v^N \in Z^s_r(\delta) \subset C([-\delta,\delta]; \mathcal{F}L^{s,r}(\mathbb{T}))$$

of FGDNLS with initial data v^N_0 . Moreover the map

$$\big(B_R, \|\cdot\|_{\mathcal{F}L^{s,r}(\mathbb{T})}\big) \longrightarrow C([-\delta,\delta]; \mathcal{F}L^{s,r}(\mathbb{T})) : \ \ v^N_0 \to v^N$$

is real analytic.

The proof essentially follows from Grünrock-Herr's [33] estimates for local well-posedness of (1.97), see [33, 48].

Lemma 1.37. *[Approximation lemma] Let $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T}), s \geq \frac{1}{2}, r \in (2,4)$ be such that $\|v_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < A$, for some $A > 0$, and let N be a large integer. Assume the solution v^N of (1.105) with initial data $v_0^N(x) = P_N v_0$ satisfies the bound*

$$\|v^N(t)\|_{\mathcal{F}L^{s,r}(\mathbb{T})} \leq A, \text{ for all } t \in [-T,T],$$

for some given $T > 0$. Then the IVP (1.97) with initial data v_0 is well-posed on $[-T,T]$ and there exists $C_0, C_1 > 0$, such that its solution $v(t)$ satisfies the estimate:

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \lesssim \exp[C_0(1+A)^{C_1}T]N^{s_1-s},$$

for all $t \in [-T,T], 0 < s_1 < s$.

For the proof see [5, 48].

Construction of Weighted Wiener Measures

We need to construct probability spaces on which we establish well-posedness almost surely. To construct these measures we will make use of the conserved quantity $\tilde{\mathcal{E}}(v)$ defined in (1.99) as well as the L^2-norm. But we cannot construct a finite measure directly using this quantity since the nonlinear part of $\tilde{\mathcal{E}}(v)$ is not bounded below and the linear part is only non-negative but not positive definite. To resolve this issue we proceed as follows. As we learned above, we use the conservation of the L^2-norm and consider instead the quantity

$$\chi_{\{\|v\|_{L^2} \leq B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy $\tilde{\mathcal{E}}(v)$, i.e.

$$\mathcal{N}(v) = -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2 \overline{v}\overline{v}_x\,dx - \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v|^2\,dx\right)\left(\int_{\mathbb{T}} |v|^4\,dx\right) + \\ + \frac{1}{\pi}\left(\int_{\mathbb{T}} |v|^2\,dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\,dx\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}} |v|^2\,dx\right)^3,$$

and B is a (suitably small) constant. Then we would like to construct the measure (with $v(x) = u(x) + iw(x)$)

$$\text{`}d\mu_\beta = Z^{-1}\chi_{\{\|v\|_{L^2} \leq B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx} \prod_{x\in\mathbb{T}} du(x)dw(x)\text{`}$$

This is a purely formal, although suggestive, expression. In particular as it will turn out $\int |u_x|^2 = \infty$, μ almost surely. We learned that one uses instead a Gaussian measure as reference measure and the weighted measure μ is constructed in two steps: first one constructs a Gaussian measure ρ as the limit of the finite-dimensional measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{\beta}{2} \sum_{|n|\le N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\le N} da_n db_n$$

where $\widehat{v}_n = a_n + ib_n$. The construction of such Gaussian measures on Hilbert spaces is a classical subject. But this time we need to realize this measure as a measure supported on a suitable Banach space, the Fourier-Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$, in view of the local well-posedness result by Grünrock-Herr. Since $\mathcal{F}L^{s,r}$ is not a Hilbert space, we need to construct ρ as a measure supported on a Banach space. This needs some extra work but it is possible by relying on Gross [31] and Kuo [42] theory of abstract Wiener spaces, (from here the name of Weighted Wiener Measures).

In particular, we prove that for $2 \le r < \infty$ and $(s-1)r < -1$: $(i, H^1, \mathcal{F}L^{s,r})$, (i=inclusion map), is an abstract Wiener space[18]. The Wiener measure ρ can be realized as a countably additive measure supported on $\mathcal{F}L^{s,r}$ and also in this case we have an exponential tail estimate. In fact

Lemma 1.38. *There exists $c > 0$ (with $c = c(s,r)$) such that*

$$\rho(\|v\|_{\mathcal{F}L^{s,r}} > K) \le e^{-cK^2}.$$

Here we assume again that v is of the form

$$v(x) = \sum_n \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{inx},$$

where $\{g_n(\omega)\}$ are independent standard complex Gaussian random variables as above. Once this measure ρ has been constructed, with a nontrivial amount of work and using also some estimates in Thomann and Tzvetkov [62], one then obtains the weighted Wiener measure μ. More precisely we define

$$R(v) := \chi_{\{\|v\|_{L^2}\le B\}} e^{-\frac{1}{2}\mathcal{N}(v)}, \qquad R_N(v) := R(v^N)$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy $\tilde{\mathcal{E}}$. Here $v^N = P_N(v)$ for some generic function v in our $\mathcal{F}L^{s,r}$ spaces.

We obtain

$$d\mu = Z^{-1} R(v) d\rho,$$

for sufficiently small B, as is the weak limit of the finite dimensional weighted Wiener measures μ_N on $\mathbb{R}^{4N+2}$ given by

[18] Note For (r,s) as above $\underbrace{s + \frac{1}{r} - \frac{1}{2}}_{=:\sigma} < \frac{1}{2}$ (recall $\mathcal{F}L^{s,r}$ scales as H^σ).

$$\begin{aligned} d\mu_N &= Z_N^{-1} R_N(v) d\rho_N \\ &= \hat{Z}_N^{-1} \chi_{\{\|\hat{v}^N\|_{L^2} \leq B\}} e^{-\frac{1}{2}(\tilde{\mathcal{E}}(\hat{v}^N) + \|\hat{v}^N\|_{L^2})} \prod_{|n| \leq N} da_n db_n \end{aligned}$$

for suitable normalizations Z_N, $\hat{Z}_N$. More precisely we have:

Lemma 1.39 (Convergence, [48]). *$R_N(v)$ converges in measure to $R(v)$.*

Moreover we have

Proposition 1.40 (Existence of weighted Wiener measure [48]).
(a) For sufficiently small $B > 0$, we have $R(v) \in L^2(d\rho)$. In particular, the weighted Wiener measure μ is a probability measure, absolutely continuous with respect to the Wiener measure ρ.
(b) We have the following tail estimate. Let $2 \leq r < \infty$ and $(s-1)r < -1$; then there exists a constant C such that

$$\mu(\|v\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-CK^2}$$

for sufficiently large $K > 0$.
(c) The finite dimension weighted Wiener measure μ_N converges weakly to μ.

We do not show the proof here but we give an example of an estimate involved in the proof instead. Recall that

$$R_N(v) := \chi_{\{\|v^N\|_{L^2} \leq B\}} e^{-\frac{1}{2}\mathcal{N}(v^N)},$$

and

$$\begin{aligned} \mathcal{N}(v) = &-\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2 \overline{v}\overline{v}_x \, dx - \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v|^2 \, dx\right)\left(\int_{\mathbb{T}} |v|^4 \, dx\right) + \\ &+ \frac{1}{\pi}\left(\int_{\mathbb{T}} |v|^2 \, dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v \overline{v}_x \, dx\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}} |v|^2 \, dx\right)^3. \end{aligned}$$

Here we concentrate on the term $X_N(v) := \int_{\mathbb{T}} v^N \overline{v}_x^N$. We have the following

Lemma 1.41. *For any $N \leq M$ and $\varepsilon > 0$ we have*

$$\|X_M(v) - X_N(v)\|_{L^4} \lesssim \frac{1}{N^{\frac{1}{2}}}.$$

Proof. We start by recalling that $v^N(\omega, x) := \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle} e^{inx}$. Then by Plancherel

$$X_N(v) = -i \sum_{|n| \leq N} n \frac{|g_n(\omega)|^2}{\langle n \rangle^2} \quad \text{and} \quad X_M(v) - X_N(v) = -i \sum_{N \leq |n| \leq M} n \frac{|g_n(\omega)|^2}{\langle n \rangle^2},$$

and

$$|X_M(v)-X_N(v)|^2 = \sum_{N\leq|n_1|,|n_2|\leq M} n_1n_2\frac{|g_{n_1}(\omega)|^2|g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\langle n_2\rangle^2} =: Y^1_{N,M}+Y^2_{N,M}+Y^3_{N,M},$$

$$\begin{aligned} Y^1_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} n_1n_2\frac{(|g_{n_1}(\omega)|^2-1)(|g_{n_2}(\omega)|^2-1)}{\langle n_1\rangle^2\langle n_2\rangle^2}\\ Y^2_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} n_1n_2\frac{(|g_{n_1}(\omega)|^2-1)+(|g_{n_2}(\omega)|^2-1)}{\langle n_1\rangle^2\langle n_2\rangle^2}\\ Y^3_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} \frac{n_1n_2}{\langle n_1\rangle^2\langle n_2\rangle^2}. \end{aligned}$$

By symmetry

$$Y^3_{N,M} = \sum_{N\leq|n_2|,|n_1|\leq M} \frac{n_1n_2}{\langle n_1\rangle^2\langle n_2\rangle^2} = 0,$$

hence

$$\|X_M(v)-X_N(v)\|^4_{L^4} \lesssim \|Y^1_{N,M}\|^2_{L^2} + \|Y^2_{N,M}\|^2_{L^2}.$$

We now proceed as in Thomann and Tzvetkov [62]: denote by

$$G_n(\omega) := |g_n(\omega)|^2 - 1$$

and note that by the definition of $g_n(\omega)$

$$\mathbb{E}[G_n(\omega)G_m(\omega)] = 0 \quad \text{for } n\neq m.$$

Since

$$|Y^1_{N,M}|^2 = \sum_{N\leq|n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1n_2n_3n_4\frac{G_{n_1}G_{n_2}G_{n_3}G_{n_4}}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

when we compute $\mathbb{E}[|Y^1_{N,M}|^2]$ the only contributions come from ($n_1=n_3$ and $n_2=n_4$), ($n_1=n_2$ and $n_3=n_4$) or ($n_2=n_3$ and $n_1=n_4$). Hence by symmetry we have:

$$\|Y^1_{N,M}\|^2_{L^2} = E[|Y^1_{N,M}|^2] \leq C \sum_{N\leq|n_1|,|n_2|\leq M} \frac{n_1^2n_2^2}{\langle n_1\rangle^4\langle n_2\rangle^4} \lesssim \frac{1}{N^2}.$$

On the other hand, since

$$|Y^2_{N,M}|^2 = \sum_{N\leq|n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1n_2n_3n_4\frac{(G_{n_1}+G_{n_2})(G_{n_3}+G_{n_4})}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

by symmetry it is enough to consider a single term of the form

$$\sum_{N\leq|n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1n_2n_3n_4 \frac{G_{n_j}G_{n_k}}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

with $1 \leq j \neq k \leq 4$, which we set without any loss of generality to be $j=1$, $k=3$. We then have

$$\|Y^2_{N,M}\|^2_{L^2} = E[|Y^2_{N,M}|^2] \leq C \sum_{N\leq|n_1|,|n_2|,|n_4|\leq M} \frac{n_1^2n_2n_4}{\langle n_1\rangle^4\langle n_2\rangle^2\langle n_4\rangle^2} = 0.$$

Analysis of the FGDNLS (1.105)

The key step now is to prove the analogue of Bourgain's Proposition 1.34 that controls the growth of solutions v^N to FGDNLS. These are the obstacles we have to face: the symplectic form associated to the periodic gauged derivative nonlinear Schrödinger equation GDNLS does not commute with Fourier modes truncation and so the truncated finite-dimensional systems are not necessarily Hamiltonian. This entails two problems:

- A mild one: one needs to show the invariance of Lebesgue measure associated to FGDNLS (1.105). ('Liouville's theorem') by hand directly .
- A more serious one and at the heart of this work: the energy $\tilde{\mathcal{E}}(v^N)$ is no longer conserved. In other words, the finite dimensional weighted Wiener measure μ_N is no longer invariant.

We show however that $\tilde{\mathcal{E}}(v^N)$ is *almost* invariant in the sense that we can control its growth in time. This idea is reminiscent of the I-method. However, in the I-method one needs to estimate the variation of the energy of *solutions to the infinite dimensional equation* at time t smoothly projected onto frequencies of size up to N. Here, one needs to control the variation of the energy $\tilde{\mathcal{E}}$ of the solution v^N to the finite dimensional approximation equation. More precisely, we have the following result:

Theorem 1.42 (Energy Growth Estimate). *Let $v^N(t)$ be a solution to* (1.105) *in $[-\delta,\delta]$, and let $K>0$ be such that $\|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)} \leq K$. Then there exists $\beta>0$ such that*

$$|\mathcal{E}(v^N(\delta)) - \mathcal{E}(v^N(0))| = \Big|\int_0^\delta \frac{d}{dt}\mathcal{E}(v^N)(t)dt\Big| \lesssim C(\delta)N^{-\beta}\max(K^6,K^8).$$

Remark 1.43. This estimate may still hold for a different choice of $X_r^{s,\frac{1}{2}}(\delta)$ norm, with $s\geq\frac{1}{2}$, $2<r<4$ so that the local well-posedness holds. On the other hand, the pair (s,r) should also be such that $(s-1)\cdot r<-1$ since this regularity is low enough to contain the support of the Wiener measure. Our choice of $s=\frac{2}{3}-$ and $r=3$ allows us to prove the energy growth estimate while satisfying both the conditions for local well-posedness and the support of the measure.

Here we only present few idea that enter in the proof, for a complete argument see [48].

We start by writing

$$
\begin{aligned}
\frac{d\tilde{\mathcal{E}}}{dt}(v^N) =& -2\mathrm{Im}\int v^N\overline{v^N v^N_x}P_N^{\perp}((v^N)^2\overline{v^N_x}) + \mathrm{Re}\int v^N\overline{v^N v^N_x}P_N^{\perp}(|v^N|^4 v^N)\\
&-2m(v^N)\mathrm{Re}\int v^N\overline{v^N v^N_x}P_N^{\perp}(|v^N|^2 v^N)\\
&+2m(v^N)\mathrm{Re}\int v^N\overline{v^N}^2 P_N^{\perp}((v^N)^2\overline{v^N_x})\\
&+m(v^N)\mathrm{Im}\int v^N\overline{v^N}^2 P_N^{\perp}(|v^N|^4 v^N)\\
&-2m(v^N)^2\mathrm{Im}\int v^N\overline{v^N}^2 P_N^{\perp}(|v^N|^2 v^N) + \ldots\ldots,
\end{aligned}
$$

The first term is the worst since it has two derivatives. Also it looks like the unfavorable structure of the nonlinearity $(v^N)^2\overline{v^N_x})$ is back! Let us now concentrate on the first term coming from the expression above. It is essentially:

$$
I_1 = \int_0^\delta \int_{\mathbb{T}} v^N\overline{v^N v^N_x}P_N^{\perp}((v^N)^2\overline{v^N_x})\,dx\,dt.
$$

We start by discussing how to absorb the rough time cut-off. Assume ϕ is any function in $X_3^{\frac{2}{3}-,\frac{1}{2}}$ such that

$$
\phi|_{[-\delta,\delta]} = v^N;
$$

then we write

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}\times\mathbb{R}} \chi_{[0,\delta]}(t)\ P_N^{\perp}((v^N)^2\partial_x\overline{v^N})\,v^N\overline{v^N v^N_x}dxdt\\
&= \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}((\chi_{[0,\delta]}\phi)^2\,\chi_{[0,\delta]}\overline{\phi_x})\ \chi_{[0,\delta]}\phi\,\chi_{[0,\delta]}\overline{\phi}\,\chi_{[0,\delta]}\overline{\phi_x}dxdt.
\end{aligned}
$$

By denoting $w := \chi_{[0,\delta]}\phi$, hence $w = P_N(w)$, we will in fact show that

$$
\begin{aligned}
|I_1| &= \left|\int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}((w)^2\partial_x\overline{w})\,w\overline{w}\overline{w_x}dxdt\right|\\
&\le C(\delta)N^{-\beta}\|w_1\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\|w_2\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\|w_3\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\\
&\times\|w_4\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\|w_5\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\|w_6\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}},
\end{aligned}
$$

where $w_1 = w_2 = w_4 = w$ and $\overline{w_3} = \overline{w_5} = \overline{w_6} = \overline{w}$. To go back to v^N then one uses the fact that for $b < b_1 < 1/2$, there exists $C'(\delta) > 0$ such that

$$
\|w\|_{X_3^{\frac{2}{3}-,b}} \le C'(\delta)\,\|\phi\|_{X_3^{\frac{2}{3}-,b_1}} \le C'(\delta)\,\|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)}
$$

where w, ϕ and v^N are as above.

We now list some of the ingredients for the proof of the estimate:

- A trilinear refinement of Bourgain's $L^6(\mathbb{T})$ Strichartz estimate: Let $u, v, w \in X^{\epsilon,\frac{1}{2}-}$ for some $\epsilon > 0$. Then

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\epsilon,\frac{1}{2}-}} \|v\|_{X^{\epsilon,\frac{1}{2}-}} \|w\|_{X^{0,\frac{1}{2}-}}$$

- Certain arithmetic identities that relate frequencies to the distance to the parabola

$$P = \{(n,\tau) \ : \ \tau = n^2\}$$

  where the solution of the linear problem lives. These estimates are important since one would like to trade derivatives, that is powers of frequencies like $|n|^\alpha$, with powers of $|\tau - n^2|$.

We notice that we can write

$$\begin{aligned} I_1 &= \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}(w^2 \partial_x \overline{w})\, w\, \overline{w \overline{w}_x} dxdt \\ &= \int \sum_{|n|>N} \left(\int_{\tau=\tau_1+\tau_2-\tau_3} \sum_{n=n_1+n_2-n_3} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(-in_3)\widehat{\overline{w}}(n_3,\tau_3)d\tau_1 d\tau_2 \right) \\ &\times \left(\int_{-\tau=\tau_4-\tau_5-\tau_6} \sum_{-n=n_4-n_5-n_6} \widehat{w}(n_4,\tau_4)\widehat{\overline{w}}(n_5,\tau_5)(-in_6)\widehat{\overline{w}}(n_6,\tau_6)d\tau_4 d\tau_5 \right) d\tau \end{aligned}$$

and from here one has

$$\begin{aligned} &\tau - n^2 - (\tau_1 - n_1^2) - (\tau_2 - n_2^2) - (\tau_3 + n_3^2) = -2(n-n_1)(n-n_2), \\ &\tau - n^2 + (\tau_4 - n_4^2) + (\tau_5 + n_5^2) + (\tau_6 + n_6^2) = -2(n+n_5)(n+n_6). \end{aligned}$$

These are the kind of relationships that we want to exploit. More precisely, if we let $\tilde{\sigma}_j := \tau_j \pm n_j^2$ we have

$$\sum_{j=1}^{6} \tilde{\sigma}_j = -2\,(\,n\,(n_1+n_2+n_5+n_6) - n_1 n_2 + n_5 n_6\,)$$

This in turn can also be rewritten using $n_1+n_2+n_3+n_4+n_5+n_6 = 0$ or $n = n_1+n_2+n_3$ and $-n = n_4+n_5+n_6$ as

$$\sum_{j=1}^{6} \tilde{\sigma}_j = 2(\,n\,(n_3+n_4) + n_1 n_2 - n_5 n_6\,).$$

Moreover, since $\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6 = 0$, by adding and subtracting n_j^2, $j = 1,\dots,6$ in the appropriate fashion, we obtain

$$\sum_{j=1}^{6} \tilde{\sigma}_j = (n_3^2 + n_5^2 + n_6^2) - (n_1^2 + n_2^2 + n_4^2).$$

We now write

$$\begin{aligned}
|I_1| &= \left| \sum_{N_i \leq N;\, i=1,\ldots 6} \int_{\mathbb{R}} \int_{\mathbb{T}} P_N^{\perp} \left(w_{N_1}\, w_{N_2}\, \partial_x \overline{w_{N_3}} \right) w_{N_4}\, \overline{w_{N_5}}\, \partial_x \overline{w_{N_6}}\, dxdt \right| \\
&= \Bigg| \sum_{N_i \leq N;\, i=1,\ldots 6} \sum_{|n| \geq N} \int_{\tau} \Bigg(\int_{\tau = \sum_{i=1}^3 \tau_i} \sum_{n = \sum_{i=1}^3 n_i} \widehat{w_{N_1}}\, \widehat{w_{N_2}}\, (in_3)\, \widehat{\overline{w_{N_3}}}\, d\tau_1 d\tau_2 \Bigg) \quad \times \\
&\quad \Bigg(\int_{-\tau = \sum_{j=4}^6 \tau_j} \sum_{-n = \sum_{j=4}^6 n_j} \widehat{w_{N_4}}\, \widehat{\overline{w_{N_5}}}\, (in_6)\, \widehat{\overline{w_{N_6}}}\, d\tau_4 d\tau_5 \Bigg) d\tau \Bigg| \\
&\leq \sum_{N \leq |n| \leq 3N} \sum_{N_i \leq N;\, i=1,\ldots 6} \int_{\tau} \Bigg(\int_{\tau = \sum_{i=1}^3 \tau_i} \sum_{n = \sum_{i=1}^3 n_i} |\widehat{w_{N_1}}| |\widehat{w_{N_2}}|\, |n_3|\, |\widehat{\overline{w_{N_3}}}|\, d\tau_1 d\tau_2 \Bigg) \quad \times \\
&\quad \Bigg(\int_{-\tau = \sum_{j=4}^6 \tau_j} \sum_{-n = \sum_{j=4}^6 n_j} |\widehat{w_{N_4}}|\, |\widehat{\overline{w_{N_5}}}|\, |n_6|\, |\widehat{\overline{w_{N_6}}}|\, d\tau_4 d\tau_5 \Bigg) d\tau.
\end{aligned}$$

Above we always think of N_j, N as dyadic; more precisely $N_j := 2^{K_j}$, $N := 2^K$ where $K_j < K$. Moreover, we denote by w_{N_j} the function such that $\widehat{w_{N_j}}(n_j) = \chi_{\{|n_j| \sim N_j\}} \widehat{w_j}(n_j)$. From the expression above we then have,

$$|n_j| \leq N, \qquad N \leq |n| \leq 3N, \qquad n = n_1 + n_2 + n_3, \quad \text{and} \quad -n = n_4 + n_5 + n_6,$$

$$N \sim \max(N_1, N_2, N_3) \sim \max(N_4, N_5, N_6),$$

We start by laying out all possible cases and organizing them according to the sizes of the two derivative terms.

I. $N_3 \sim N$, $N_6 \sim N$
II. $N_3 \sim N$ and $N_6 \ll N$
III. $N_6 \sim N$ and $N_3 \ll N$
IV. $N_3 \ll N$; $N_6 \ll N$

Now we subdivide into all subcases in each situation and group them according to how many low frequencies (ie. $N_j \ll N$) we have overall.

IA. $N_3 \sim N$, $N_6 \sim N$ and 4 lows: $N_1, N_2, N_4, N_5 \ll N$
IB. $N_3 \sim N$, $N_6 \sim N$ and 3 lows
- (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$
- (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$
- (iii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$
- (iv) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$

IC. $N_3 \sim N$, $N_6 \sim N$ and 2 lows
- (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$
- (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$
- (vi) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$

ID. $N_3 \sim N$, $N_6 \sim N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IE. $N_3 \sim N$, $N_6 \sim N$ and $N_1, N_2, N_4, N_5 \sim N$

IIA. $N_3 \sim N$ and $N_6 \ll N$ and 3 lows
- (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$
- (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$

IIB. $N_3 \sim N$ and $N_6 \ll N$ and 2 lows
- (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$
- (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIC. $N_3 \sim N$ and $N_6 \ll N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IID. $N_3 \sim N$ and $N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IIIA. $N_6 \sim N$ and $N_3 \ll N$ and 3 lows
- (i) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$
- (ii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$

IIIB. $N_6 \sim N$ and $N_3 \ll N$ and 2 lows
- (i) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$
- (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIIC. $N_6 \sim N$ and $N_3 \ll N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IIID. $N_6 \sim N$ and $N_3 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IVA. $N_3 \ll N, N_6 \ll N$ and 2 lows
- (i) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (ii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iii) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (iv) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IVB. $N_3 \ll N, N_6 \ll N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IVC. $N_3 \ll N, N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$.

The following lemma is fundamental

Lemma 1.44. *If* $0 < \beta < 2$, *then*

$$\|J_\beta w_M\|_{X^{0,\rho}} \lesssim C_T\, A(\beta, M)^{\frac{1}{6}} M^{\rho\beta+} \|w_M\|_{X_3^{0,\frac{1}{6}}},$$

where

$$(i) \quad supp\ w_M(\cdot, x) \subset [-T, T] \qquad (x \in \mathbb{T}).$$
$$(ii)\ \widehat{J_\beta w_M}(\tau, n) = \chi_{\{|n|\sim M\}} \chi_{\{|\tau - n^2| \le M^\beta\}} |\widehat{w_M}(\tau, n)|.$$

Here, if

$$S(\tau, M, \beta) := \{n \in \mathbb{Z} : |n| \sim M \text{ and } |\tau - n^2| \le M^\beta\}$$

and $|S|$ represents the counting measure of the set S, then one can show that

$$A(M, \beta) := \sup_\tau |S(\tau, M, \beta)| \le 1 + M^{\beta - 1}.$$

We do not prove the lemma (see [48]), we only show a simple but useful counting argument. If $S := S(\tau, M, \beta) \neq \emptyset$, then there exists $n_0 \in S$ and hence

$$\begin{aligned}|S| &\le 1 + |\{l \in \mathbb{Z}/ |\, n_0 + l\,| \sim M,\ |\tau - (n_0 + l)^2| \le M^\beta\}| \\ &\le 1 + |\{l \in \mathbb{Z}\, /\, |l| \le M,\ |2n_0 l + l^2| \lesssim M^\beta\}|.\end{aligned}$$

Now we note that $|2n_0 l + l^2| = |(l + n_0)^2 - n_0^2| \lesssim M^\beta$ if and only if

$$-CM^\beta + n_0^2 \le (l + n_0)^2 \le n_0^2 + CM^\beta.$$

Hence we need $|\,l\,| \le M$ to satisfy

$$-\sqrt{n_0^2 + CM^\beta} \ \le\ (l + n_0)\ \le\ \sqrt{n_0^2 + CM^\beta},$$
$$(l + n_0)\ \ge\ \sqrt{n_0^2 - CM^\beta} \quad \text{and} \quad (l + n_0)\ \le\ -\sqrt{n_0^2 - CM^\beta}.$$

In other words, we need to know the size of

$$[-\sqrt{n_0^2 + CM^\beta}, -\sqrt{n_0^2 - CM^\beta}] \cup [\sqrt{n_0^2 - CM^\beta}, \sqrt{n_0^2 + CM^\beta}]$$

which is of order $\frac{M^\beta}{|n_0|}$. Hence, since $|n_0| \sim M$, we have that $|S| \le 1 + M^{\beta-1}$, as claimed.

Armed with the energy growth estimate of Theorem 1.42 we count on the almost invariance of the finite-dimensional measure μ_N under the flow of (1.105) to control the growth of its solutions (our analogue of Bourgain's Proposition 1.34).

Proposition 1.45. *[Growth of solutions to FGDNLS] For any given $T > 0$ and $\varepsilon > 0$ there exists an integer $N_0 = N_0(T, \varepsilon)$ and sets $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T) \subset \mathbb{R}^{2N+2}$ such that for $N > N_0$*

(a) $\mu_N\left(\widetilde{\Omega}_N\right) \ge 1 - \varepsilon$.

(b) For any initial condition $v_0^N \in \widetilde{\Omega}_N$, FGDNLS (1.105) is well-posed on $[-T, T]$ and its solution $v^N(t)$ satisfies the bound

$$\sup_{|t|\le T} \|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

Combining the Approximation Lemma 1.37 of v by v^N with Proposition 1.45 on the growth of solutions to FGDNLS (1.105) we can prove a similar result for solutions v to GDNLS (1.97):

Proposition 1.46 ('Almost almost' sure GWP for GDNLS (1.97)). *For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon, T)$ such that*

(a) $\mu\left(\Omega(\varepsilon, T)\right) \ge 1 - \varepsilon$.

(b) For any initial condition $v_0 \in \Omega(\varepsilon, T)$ the IVP of (1.97) is well-posed on $[-T, T]$ with the bound

$$\sup_{|t|\le T} \|v(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

As in Bourgain's argument, we then have:

Theorem 1.47. *[Almost sure global well-posedness of GDNLS (1.97)] There exists a set Ω, $\mu(\Omega^c) = 0$ such that for every $v_0 \in \Omega$ the IVP of GDNLS with initial data v_0 is globally well-posed.*

and

Theorem 1.48. *[Invariance of μ] The measure μ is invariant under the flow $\Phi(t)$ of GDNLS.*

For the proof of Theorems 1.47 and 1.48 see [48].

The last step is going back to the original DNLS (1.94). Recall that if μ is a measure on Ω and $\mathcal{G}^{-1}:\Omega\to\Omega$ measurable, the measure $\nu=\mu\circ\mathcal{G}$ is defined

$$\nu(A)\;:=\;\mu(\mathcal{G}(A))\;=\;\mu(\{v\;:\;\mathcal{G}^{-1}(v)\in A\})\,. \tag{1.106}$$

for all measurable sets A or, equivalently, for integrable F by

$$\int F d\nu=\int F\circ\varphi\, d\mu.$$

By pulling back the gauge, it follows easily from Theorem 1.47 that

Theorem 1.49 (Almost sure global well-posedness of DNLS (1.94)**).** *There exists a subset Σ of the space $\mathcal{F}L^{\frac{2}{3}-,3}$ with $\nu(\Sigma^c)=0$ such that for every $u_0\in\Sigma$ the IVP DNLS* (1.94) *with initial data u_0 is globally well-posed.*

Proof. Let Ω be the set of full μ measure given in Theorem 1.47 and let $\Sigma=G^{-1}(\Omega)$. Note that Σ is a set of full ν-measure by (1.106). For $v_0\in\Omega$ the IVP GDNLS (1.97) with initial data v_0 is globally well-posed. Hence since the map $\mathcal{G}:C([-T,T];\mathcal{F}L^{s,r})\to C([-T,T];\mathcal{F}L^{s,r})$ is a homeomorphism, if $s>\frac{1}{2}-\frac{1}{r}$ where $2<r<\infty$, the initial value problem associated to DNLS (1.94) with initial data $u_0=G^{-1}(v_0)$ is also globally well-posed.

Finally one shows that the measure ν is invariant under the flow map of DNLS (1.94), [48]

Theorem 1.50 (Invariance of measure under the DNLS flow (1.94)**).** *The measure $\nu=\mu\circ\mathcal{G}$ is invariant under the DNLS flow* (1.94).

What is $\nu=\mu\circ\mathcal{G}$ really? Is this absolutely continuous with respect to the measure that can be naturally constructed for DNLS by using its energy E,

$$\begin{aligned} E(u)\;&=\;\int_{\mathbb{T}}|u_x|^2\,dx+\frac{3}{2}\mathrm{Im}\int_{\mathbb{T}}u^2\overline{uu_x}\,dx+\frac{1}{2}\int_{\mathbb{T}}|u|^6\,dx\\ &=:\int_{\mathbb{T}}|u_x|^2\,dx+\mathcal{K}(u)\end{aligned}$$

as done by Thomann-Tzevtkov [62]? We know ν is invariant and that the ungauged DNLS equation (1.94) is globally well-posed a.s with respect to ν. It is easy to see that treating the weight is easy thanks to a direct calculation. The problem is un-gauging the Gaussian measure ρ. We can ask the following question: What is $\tilde{\rho}:=\rho\circ\mathcal{G}$? Is its restriction to a sufficiently small ball in L^2 absolutely continuous with respect to ρ? If so, what is its Radon-Nikodym derivative?

1.7 Gauss measures and gauge transformations

In order to finish this step one should stop thinking about PDE, solutions v viewed as infinite dimension vectors of Fourier modes, etc. Instead one should start thinking about (periodic) complex Brownian paths in $\mathbb{T}$ (Brownian bridge) solving certain stochastic equations and a certain transformation acting on these paths. This because there are theorems, such as Girsanov's Theorem 1.51 below, that are able to connect how a certain law changes under a transformation by analyzing the drift term associated to the transformed stochastic equation. What follows can be found in full details in [51]. The reader should be warned that here we only give a very rough idea of the argument.

We recall that to un-gauge we need to define

$$\mathcal{G}^{-1}(v)(x) := \exp(iJ(v))\, v(x)$$

where

$$J(v)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x |v(y)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2(\mathbb{T})}\, dy\, d\theta.$$

It will be important later that $J(v)(x) = J(|v|)(x)$. Then, let v be a complex Brownian motion that satisfies the stochastic equation

$$dv(x) = \underbrace{dB(x)}_{\text{Brownian motion}} + \underbrace{b(x)dx}_{\text{drift terms}}. \tag{1.107}$$

By Ito's calculus, since $\exp(iJ(v))$ is differentiable, we have:

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\, dv + iv\exp(iJ(v))\left(|v(x)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2}\right) dx + \dots. \tag{1.108}$$

Substituting dv from (1.107) above one has

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\, [dB(x) + a(v,x,\omega))\, dx] + \dots$$

where

$$a(v,x,\omega) = iv\ \left(|v(x)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2}\right). \tag{1.109}$$

What could help? Certainly the fact that $\exp(iJ(v))$ is a unitary operator and that one can prove Novikov's condition:

$$E\left[\exp\left(\frac{1}{2}\int a^2(v,x,\omega)dx\right)\right] < \infty. \tag{1.110}$$

In fact, this last condition looks exactly like what we need for the following theorem:

Theorem 1.51 (Girsanov [49]). *If we change the drift coefficient of a given Ito process in an appropriate way, see* (1.109)*, then the law of the process will not change dramatically. In fact the new process law will be absolutely continuous with respect to the law of the original process and we can compute explicitly the Radon-Nikodym derivative.*

Unfortunately, though, Girsanov's theorem doesn't save the day.... at least not immediately. If one reads the theorem carefully one realizes that an important condition is that $a(v, x, \omega)$ is *non anticipative*, in the sense that it only depends on the BM v up to "time" x and not further. This unfortunately is not true in our case. The new drift term $a(v, x, \omega)$ involves the L^2 norm of $v(x)$, see (1.109), and hence it is *anticipative*. A different strategy is needed and conformal invariance of complex BM comes to the rescue.

We use the well known fact that if $W(t) = W_1(t) + iW_2(t)$ is a complex Brownian motion, and if ϕ is an analytic function then $Z = \phi(W)$ is, after a suitable time change, again a complex Brownian motion[19], [49]. For $Z(t) = \exp(W(s))$ the time change is given by

$$t = t(s) = \int_0^s |e^{W(r)}|^2 dr, \qquad \frac{dt}{ds} = |e^{W(s)}|^2,$$

equivalently

$$s(t) = \int_0^t \frac{dr}{|Z(r)|^2}, \qquad \frac{ds}{dt} = \frac{1}{|Z(t)|^2}.$$

We are interested on $Z(t)$ for the interval $0 \leq t \leq 1$ and thus we introduce the stopping time

$$\mathcal{S} = \inf\left\{ s\,;\ \int_0^s |e^{W(r)}|^2 dr = 1 \right\}$$

and remark the important fact that the stopping time $\mathcal{S}$ depends only on the real part $W_1(s)$ of $W(s)$ (or equivalently only $|Z|$). If we write $Z(t)$ in polar coordinate $Z(t) = |Z(t)|e^{i\Theta(t)}$, we have

$$W(s) = W_1(s) + iW_2(s) = \log|Z(t(s)| + i\Theta(t(s))$$

and W_1 and W_2 are real independent Brownian motions. If we define

$$\begin{aligned} \tilde{W}(s) &:= W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(|Z|)(r)dr\right] \\ &= W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(e^{W_1})(r)dr,\right] \end{aligned}$$

and recall that in our case, essentially

$$h(|Z|)(\cdot) = |Z(\cdot)|^2 - \|Z\|^2_{L^2}.$$

[19] In what follows one should think of $Z(t)$ to play the role of our complex BM $v(x)$.

We then have

$$e^{\tilde{W}(s)} = \tilde{Z}(t(s)) = \mathcal{G}^{-1}(Z)(t(s)).$$

In terms of W, the gauge transformation is now easy to understand: it gives a complex process in which the real part is left unchanged and the imaginary part is translated by the function $J(Z)(t(s))$ which depends only on the real part (ie. on $|Z|$, which has been fixed) and in that sense is deterministic. It is now possible to use Cameron-Martin-Girsanov's theorem [16, 49] only for the law of the imaginary part and conclude the proof. Then, if η denotes the probability distribution of W and $\tilde{\eta}$ the distribution of $\tilde{W}$, we have the absolute continuity of $\tilde{\eta}$ and η whence the absolute continuity between $\tilde{\rho}$ and ρ follows with the *same Radon-Nikodym derivative* (re-expressed back in terms of t). All in all, we can prove that our ungauged measure ν is in fact essentially (up to normalizing constants) of the form

$$d\nu(u) = \chi_{\|u\|_{L^2} \leq B} e^{-\mathcal{K}(u)} d\rho,$$

the weighted Wiener measure associated to DNLS (constructed by Thomann-Tzvetkov [62]). In particular, we prove its invariance.

Remark 1.52. The sketch of the argument above needs to be done carefully for complex Brownian bridges (periodic BM) by conditioning properly, see [51].

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