DISPERSIVE EQUATIONS AND THEIR ROLE BEYOND PDE

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ABSTRACT. Arguably the star in the family of dispersive equations is the Schrödinger equation. Among many mathematicians and physicists it is regarded as fundamental, in particular to understand complex phenomena in quantum mechanics.

But not many people may know that this equation, when defined on tori for example, has a very reach and more abstract structure that touches several fields of mathematics, among which analytic number theory, symplectic geometry, probability and dynamical systems.

In this talk I will illustrate in the simplest possible way how all these different aspects of a unique equation have a life of their own while interacting with each other to assemble a beautiful and subtle picture. This picture is not yet completely well understood and many questions and open problems are there ready to be solved by a new generation of mathematicians.

1. INTRODUCTION

In these notes I would like to collect some old and new results addressing very different mathematical aspects related to semilinear periodic Schrödinger equations in low dimensions. In doing so I will present some open problems that often go behind the field of partial differential equations and touches upon analytic number theory, probability, symplectic geometry and dynamical systems.

After the introduction in Section 1 I will set up the stage in Section 2. I will start Section 3 I will start with a (now classical) Strichartz inequality for the periodic linear Schrödinger equation in two dimensions due to Bourgain. I will continue with some results on local and global well-posedness for certain nonlinear Schrödinger equations.

In Section 4 I will elaborate on the growth in time of high order Sobolev norms for the global flow, whenever it exists. I will explain how the estimate of this growth could give some information on how the frequency profile of a certain wave solution could move from low to high frequencies while maintaining constant mass and energy (*forward cascade.*) I will present two results for the defocusing, cubic, periodic, two dimensional Schrödinger equation: the first is a polynomial upper bound in time for Sobolev norms of a global generic solution; the second is a weak growth result, namely that after fixing a small constant δ and a large one K, one can find a certain solution that at time zero is as small[1] as δ and at a certain time far in the future is as big as K.

In Section 5 I will use certain periodic Schrödinger equations as examples of infinite dimension Hamiltonian systems and for them I will present some old and recent results that are generalizations of fine dimensions ones. As a first example I will consider the cubic periodic defocusing NLS and I will recall the squeezing theorem due to Bourgain. Next I will introduce the concept of Gibbs measures associated to periodic semilinear Schrödinger equations in one dimension. These measure already proposed by Lebowitz, Rose and Speer

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[1]In terms of a fixed Sobolev norm.

were later proved to be invariant by Bourgain who also used this invariance to show global well-posedness at a level in which conservation laws are not available. Of course in this case global well-posedness should be understood as an almost sure result. I will then introduce the periodic derivative nonlinear Schrödinger (DNLS) equation. This is an integrable system, that also can be viewed as an Hamiltonian system. Proving that it is globally well-posed for rough data is very challenging. In fact in order to be able to use certain estimates one needs to apply a gauge transformation to the equation. Moreover even for the gauged equation local well-posedness can be obtained via a fixed point argument only on certain spaces that are of type l^p, not necessarily $p = 2$, with respect to frequency variables. Because of this when later one wants to introduce a Gibbs measure, which is in turn related to the Gaussian measure defined on Sobolev spaces $H^s, s < \frac{1}{2}$, one needs to generalize the definition and take advantage of the more abstract Wiener theory. In spite of several obstacles that one needs to overcome in order to apply a variant of Bourgain's argument, one still obtains for the gauged DNLS problem an almost surely global well-posedness result. Of course at the end one needs to "un-gauge" and I will show how a purely probabilistic argument will translate the almost surely global well-posedness for the gauged DNLS into a similar one for the original derivative nonlinear Schrödinger equation.

2. Setting up the stage

The objects of study in these notes is mainly the semilinear Schrödinger (NLS) initial value problems (IVP)

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $p > 1$, $u : \mathbb{R} \times \mathbb{T}^n \to \mathbb{C}$, and $\mathbb{T}^n$ is a n-dimensional torus[2]. We observe right away that (1) admits two conservation laws

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx = H(u_0) \tag{2}$$

called the Hamiltonian and

$$M(u(t)) = \int |u|^2(x,t)\,dx = M(u_0) \tag{3}$$

called the mass.

Schrödinger equations are classified as *dispersive* partial differential equations and the justification for this name comes from the fact that if no boundary conditions are imposed their solutions tend to be waves which spread out spatially. A simple and complete mathematical characterization of the word *dispersion* is given to us for example by R. Palais in [41].

It is probably common knowledge that dispersive equations are proposed as models of certain wave phenomena that occur in nature. But it turned out that some of these equations appear also in more abstract mathematical areas such as algebraic geometry [29] and they are found to possess surprisingly beautiful structures. Certainly I am not in the position to discuss this part of mathematics here, but nevertheless I hope I will be able to give a glimpse of various connections of these equations with other areas of mathematics.

The interesting aspect of dispersive equations, Schrödinger equations in particular, is that in later times their solutions do not acquire extra smoothness and neither remain

[2]Later we will distinguish between a rational and an irrational torus.

compact if the initial profiles were. In particular, since we will impose periodic boundary conditions, dispersion will be extremely weak. All this will make our analysis more difficult, but also more interesting.

Probably the most standard questions that one may want to ask about an IVP such as (1), since it does model physical phenomena, are existence of solutions, stability, time-asymptotic properties of solutions, blow up etc. Until recently these questions were addressed in a very deterministic way and I will report on some of these results in Sections 2, 3 and 4. In recent years there has been an increasing interest on addressing these questions using a natural probabilistic approach, this is some of the content of the remaining Section 5. The set up for this probabilistic approach is based on viewing (1) as an infinite dimensional Hamiltonian system. This is done by rewriting the equation as an Hamiltonian systems for the Fourier coefficients of the solutions to (1). Using this structure one can then formally define an invariant measure [34] acting on the infinite dimensional space given by the vectors of Fourier coefficients. This measure, proved to be invariant [5], is able to select data in rough spaces that can be evolved globally in time even when blow up may occur and in so doing gives what we call an almost surely global well-posedness.

The infinite dimensional Hamiltonian structure that we can recognize for some NLS equations, in some cases can be also equipped with a symplectic structure. Then the natural question is whether one may be able to extend fundamental concepts such a *capacity* or prove results such as Gromov's non-squeezing theorem in this infinite dimensional context, [6, 21, 32]. In these notes I will recall one of such results, see Theorem 5.13, but much more needs to be studied and discovered in this area.

It is clear by now that when possible, a strong deterministic and probabilistic approach to the study of an IVP such as (1) is certainly bound to generate not just some abstract and beautiful mathematics, but also a deeper understanding of the physical phenomena that semilinear Schrödinger equations represent.

2.1. **Notation.** Throughout these notes we use C to denote various constants. If C depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C_{\|u_0\|_2}$ will depend on $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is an absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a+\varepsilon$ and $a-\varepsilon$, for some $0<\varepsilon \ll 1$.

Finally, since we will be making heavy use of Fourier transforms, we recall here that $\hat{f}$ will usually denote the Fourier transform of f with respect to the space variables and when there is no confusion we use the hat notation even when we take Fourier transform also with respect to the time variable. In general though we will use the notation $\tilde{u}$ if we want to emphasize that we take the Fourier transform of a function $u(t,x)$ in both space and time variables.

3. Periodic Strichartz estimates

Let's start with the classical result of existence, uniqueness and stability of solutions. for an IVP. It is not hard to understand that these results strongly depend on the regularity one asks for the solutions themselves and the given data. So we first have to decide how we "measure" the regularity of function. The most common way of doing so is by deciding where the weak derivatives of the function "live". Most of the times we assume that the data are in Sobolev spaces H^s. In more sophisticated instances one may need to replace Sobolev spaces with different ones, like Besov spaces, Hölder spaces, and so on.

Since we will be dealing with functions that have a time variable we will often need mixed norm spaces, so for example, we may need that $f \in L^p_x L^q_t$, that is $\|(\|f(x,t)\|_{L^q_t})\|_{L^p_x} < \infty$. Finally, for a fixed interval of time $[0,T]$ and a Banach space of functions Z, we denote with $C([0,T],Z)$ the space of continuous maps from $[0,T]$ to Z.

We are now ready to give the definition of well-posedness for the IVP (1). We start with the linear Schrödinger IVP

$$\begin{cases} iv_t + \Delta v = 0, \\ v(x,0) = u_0(x). \end{cases} \tag{4}$$

The solution $v(x,t) =: S(t)u_0(x)$ of this IVP will be studied below, for now we will use it to write the solution to (1).

Definition 3.1. *We say that the IVP* (1) *is* locally well-posed (l.w.p) *in* $H^s(\mathbb{R}^n)$ *if for any ball* B *in the space* $H^s(\mathbb{R}^n)$ *there exist a time* T *and a Banach space of functions* $X \subset L^\infty([-T,T],H^s(\mathbb{R}^n))$ *such that for each initial data* $u_0 \in B$ *there exists a unique solution* $u \in X \cap C([-T,T],H^s(\mathbb{R}^n))$ *for the integral equation*[3]

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')|u|^{p-1}u(t'))\,dt'. \tag{5}$$

Furthermore the map $u_0 \to u$ *is continuous as a map from* H^s *into* $C([-T,T],H^s(\mathbb{R}^n))$*. If uniqueness is obtained in* $C([-T,T],H^s(\mathbb{R}^n))$*, then we say that local well-posedness is "unconditional".*

If Definition 3.1 holds for all $T > 0$ then we say that the IVP is *globally well-posed* (g.w.p).

Remark 3.2. *The intervals of time are symmetric about the origin because the problems that we study here are all time reversible (i.e. if* $u(x,t)$ *is a solution, then so is* $-u(x,-t)$*).*

Usually the way one proves well-posedness, at least locally, is by defining an operator

$$Lv = S(t)u_0 + c\int_0^t S(t-t')|v|^{p-1}v(t'))\,dt'$$

and then showing that in a certain space of functions X one has a fixed point and as a consequence a solution according to (5). The hard part is to decide what space X could work. The general idea is to show strong estimates[4] for the solution $S(t)u_0$ of the linear problem (4), identify the space X from these estimates and expect that the solution u also satisfies them at least when through (5) one can show that u is a perturbation of the linear problem. This kind of argument usually works in so called subcritical regimes[5] and for short times; for long times and critical regimes the situation could be much more complicated.

Remark 3.3. *Our notion of global well-posedness does not require that* $\|u(t)\|_{H^s(\mathbb{R}^n)}$ *remains uniformly bounded in time. In fact, unless* $s = 0,1$ *and one can use the conservation of mass or energy, it is not a triviality to show such an uniform bound. This can be obtained as a consequence of scattering, when scattering is available. In general this is a question related to* weak turbulence theory *and we will address it more in details in Section 4.*

[3]Note that (1) is equivalent to (5) via the Duhamel principle when enough regularity is assumed.

[4]For example Strichartz estimates in Section 3.

[5]If we write $H(u(t)) = K(u(t)) + \lambda P(u(t))$, where $K(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx$ is the kinetic energy and $P(u(t)) = \frac{2}{p+1}\int |u(t,x)|^{p+1}\,dx$ is the potential one, then the energy subcritical regime is when the kinetic energy is stronger than the potential one.

We are now ready to introduce some of the most important estimates relative to the solution $S(t)u_0$ to the linear Schrödinger IVP (4). This solution is easily computable by taking Fourier transform. In fact for each fixed frequency ξ problem (4) transforms into the ODE

$$\begin{cases} i\hat{v}_t(t,\xi) - |\xi|^2\hat{v}(t,\xi) = 0, \\ \hat{v}(\xi,0) = \hat{u}_0(\xi) \end{cases} \tag{6}$$

and we can write its solution as

$$\hat{v}(t,\xi) = e^{-i|\xi|^2 t}\hat{u}_0(\xi).$$

We observe that what we just did works both in $\mathbb{R}^n$ and $\mathbb{T}^n$.

3.1. **Strichartz estimates in $\mathbb{R}^n$.** If we define, in the distributional sense,

$$K_t(x) = \frac{1}{(\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}},$$

we then have

$$S(t)u_0(x) = e^{it\Delta}u_0(x) = u_0 \star K_t(x) = \frac{1}{(\pi i t)^{n/2}} \int e^{i\frac{|x-y|^2}{2t}} u_0(y)\,dy. \tag{7}$$

As mentioned already

$$\widehat{S(t)u_0}(\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi), \tag{8}$$

and from here $S(t)u_0(x)$ can be interpreted as the adjoint of the Fourier restriction operator on the paraboloid $P = \{(\xi, |\xi|^2) \text{ for } \xi \in \mathbb{R}^n\}$. This remark, strictly linked to (7) and (8), can be used to prove a variety of very deep estimates for $S(t)u_0$, see for example [15, 45]. From (7) we immediately have the so called *dispersive estimate*

$$\|S(t)u_0\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\|u_0\|_{L^1}. \tag{9}$$

From (8) instead we have the conservation of the homogeneous Sobolev norms[6]

$$\|S(t)u_0\|_{\dot{H}^s} = \|u_0\|_{\dot{H}^s}, \tag{10}$$

for all $s \in \mathbb{R}$. Interpolating (9) with (10) when $s = 0$ and using a so called TT^* argument one can prove the famous Strichartz estimates summarized in the following theorem:

Theorem 3.4. *[Strichartz Estimates in $\mathbb{R}^n$] Fix $n \geq 1$. We call a pair (q,r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q,r,n) \neq (2,\infty,2)$. Then for any admissible exponents (q,r) and $(\tilde{q},\tilde{r})$ we have the homogeneous Strichartz estimate*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{11}$$

and the inhomogeneous Strichartz estimate

$$\left\| \int_0^t S(t-t')F(t')\,dt' \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{12}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

See [30] and [47] for some concise proofs, and [15] for a complete list of authors who contributed to the final version of this theorem.

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[6]We will see later that the L^2 norm is conserved also for the nonlinear problem (1).

3.2. **Strichartz estimates in $\mathbb{T}^n$.** In this section we will see how essential is the assumption that $\mathbb{T}^n$ is a rational torus[7] in order to be able to prove sharp Strichartz estimates. The conjecture is that for irrational tori one should be able to prove similar estimates, if not better in some cases, but for now the best available results are due to Bourgain in [9, 10]. In a sense irrational tori should generate some sort of weak dispersion since the reflections of the wave solutions through periodic boundary conditions, with periods irrational with respect to each other, should interact less in the nonlinearity. As for now there are no results of this type in the literature.

Assume that $c_i > 0$, $i = 1, ..., n$ are the periods with respect to each coordinate. In the periodic case one cannot expect the range of admissible pairs (q, r) as in Theorem 3.4. We concentrate on the pairs $q = r$, that is $q = \frac{2(n+2)}{n}$. There is the following conjecture:

Conjecture 3.1. *Assume that $\mathbb{T}^n$ is a rational torus and the support of $\hat{\phi}_N$ is in the ball $B_N(0) = \{|n| \lesssim N\}$. Write*

$$S(t)\phi_N(x) = \sum_{k\in\mathbb{Z}^n, |k|\sim N} a_k e^{i(\langle x,k\rangle - \gamma(k)t)},$$

where (a_k) are the Fourier coefficients of ϕ_N and

$$\gamma(k) = \sum_{i=1}^{n} c_i k_i^2. \tag{13}$$

If the torus is rational we can assume without loss of generality that $c_i \in \mathbb{N}$. Then

$$\|S(t)\phi_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q \|\phi_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n} \tag{14}$$

$$\|S(t)\phi_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \ll N^\epsilon \|\phi_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q = \frac{2(n+2)}{n} \tag{15}$$

$$\|S(t)\phi_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\frac{n}{2}-\frac{n+2}{q}} \|\phi_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n} \tag{16}$$

For a partial resolution of the conjecture see [4]. We present Bourgain's argument for $n = 2$, $q = 4$ below to show how the rationality of the torus comes into play.

Proof. In this proof we restrict further to the case when $c_i = 1$ for $i = 1, ..., n$. Then

$$\left\| \sum_{|k|\le N} a_n e^{i(\langle x,k\rangle - |n|^2 t)} \right\|^4_{L^4([0,1]\times\mathbb{T}^2)} = \left\| \left[\sum_{|k|\le N} a_k e^{i(\langle x,k\rangle - |k|^2 t)} \right]^2 \right\|^2_{L^2([0,1]\times\mathbb{T}^2)} = \sum_{k,m} |b_{k,m}|^2,$$

where

$$b_{k,m} = \sum_{k=k_1+k_2; m=|k_1|^2+|k_2|^2, |k_i|\le N, i=1,2} a_{k_1} a_{k_2}$$

since

$$\begin{aligned} \left[\sum_{|k|\le N} a_k e^{i(\langle x,k\rangle - |n|^2 t)} \right]^2 &= \sum_{|n_1|\le N, |n_2|\le N} a_{k_1} a_{k_2} e^{i(\langle x,(k_1+k_2)\rangle - (|k_1|^2+|k_2|^2)t)} \\ &= \sum_{k,m} b_{k,m} e^{i(\langle x,k\rangle + mt)}. \end{aligned}$$

[7]For us a torus is irrational if there are at least two coordinates for which the ratio of their periods is irrational.

Now it is easy to see that

$$\|S(t)\phi_N\|^4_{L^4_tL^4_x([0,1]\times\mathbb{T}^2)} \sim \sum_{k,m}|b_{k,m}|^2 \lesssim \sup_{|k|\lesssim N,|m|\lesssim N^2} \#M(k,m)\|(a_n)\|^4_{l^2}, \tag{17}$$

where

$$\#M(k,m) = \#\{(k_1\in\mathbb{Z}^2\ /\ 2m-|k|^2 = |k-2k_1|^2\} = \#\{(z\in\mathbb{Z}^2\ /\ 2m-|k|^2=|z|^2\}.$$

If $2m-|k|^2<0$ there are no points in $M(k,m)$, and if $R^2 := 2m-|k|^2\geq 0$, there are at most $\exp C\frac{\log R}{\log\log R}$ many points on the circle of radius R [26], and since $R^2\leq N^2$, using (17) we obtain

$$\|S(t)\phi_N\|_{L^4_tL^4_x([0,1]\times\mathbb{T}^2)} \lesssim N^\epsilon\|\phi_N\|_{L^2}, \tag{18}$$

for all $\epsilon>0$. □

Remark 3.5. *Thanks to a very precise translation invariance in the frequency space for $S(t)u$, estimate (18) holds also when the support of ϕ_N is on a ball of radius N centered in an arbitrary point $z_0\in\mathbb{Z}^2$.*

In order to set up a fixed point theorem to prove well-posedness one defines $X^{s,b}$ spaces, introduced in this context by Bourgain [4]. The norms in these spaces are defined for $s,b\in\mathbb{R}$ as:

$$\|u\|_{X^{s,b}(\mathbb{T}^2\times\mathbb{R})} := \Big(\sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}}|\widetilde{u}(n,\tau)|^2\langle n\rangle^{2s}\langle\tau+|n|^2\rangle^{2b}d\tau\Big)^{\frac12},$$

One can immediately see that these spaces are measuring the regularity of a function with respect to certain parabolic coordinates, this to reflect the fact that linear Schrödinger solutions live on parabolas. Having defined the spaces one wants to relate their norms to certain $L^q_tL^p_x$ norms that are typical of Strichartz estimates as proved above in a special case. A key estimate, proved in [4], is

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0+,\frac12+}}. \tag{19}$$

This is proved by viewing u as sum of components supported on paraboloids that are at distance one from each other, using (18) on each of them and then reassembling the estimates using the weight $\langle\tau+|n|^2\rangle^{2b}$. An additional estimate is:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{\frac12+,\frac14+}}. \tag{20}$$

The estimate (20) is a consequence of the following lemma [8].

Lemma 3.6. *Suppose that Q is a ball in $\mathbb{Z}^2$ of radius N and center z_0. Suppose that u satisfies supp $\widehat{u}\subseteq Q$. Then*

$$\|u\|_{L^4_{t,x}} \lesssim N^{\frac12}\|u\|_{X^{0,\frac14+}}. \tag{21}$$

Lemma 3.6 is proved in [8] by using Hausdorff-Young and Hölder's inequalities. We omit the details. We can now interpolate between (19) and (20) to deduce:

Lemma 3.7. *Suppose that u is as in the assumptions of Lemma 3.6, and suppose that $b_1,s_1\in\mathbb{R}$ satisfy $\frac14<b_1<\frac12+, s_1>1-2b_1$. Then*

$$\|u\|_{L^4_{t,x}} \lesssim N^{s_1}\|u\|_{X^{0,b_1}}. \tag{22}$$

Lemma 3.7 can then be used to prove local well-posedness for the cubic NLS in $\mathbb{T}^2$ in H^s, $s > 0$. One in fact can set up a fixed point argument in the space $X^{s,b}$, $s > 0$, $b \sim \frac{1}{2}$. The key point is that the problem at hand has a cubic nonlinearity which by duality forces us to consider a product of four functions in L^1. This translates into estimating L^4 norms which via (22) are related back to the space $X^{s,b}$. In the proof one shows that the interval of time $[-T, T]$ suitable for a fixed point argument is such that

$$T \sim \|u_0\|_{H^s}^{-\alpha}, \tag{23}$$

for some $\alpha > 0$. As a consequence, the defocusing, cubic, periodic NLS problem (1) can be proved to be globally well-posed in $H^s, s \geq 1$ thanks to (23) and the conservation of the Hamiltonian (2). See [4, 8].

4. Growth of Sobolev norms and energy transfer to high frequencies

We consider the cubic, defocusing, periodic (rational) NLS initial value problem:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \; x \in \mathbb{T}^2 \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}^2), \; s > 1. \end{cases} \tag{24}$$

From Section 3 we know that (24) is globally well-posed in $H^s, s \geq 1$. Hence, it makes sense to analyze the behavior of $\|u(t)\|_{H^s}$. But as we will discuss later this estimate is related to an important physical phenomenon: energy transfer to higher modes or forward cascade. We will elaborate more on this below.

Theorem 4.1 (Bound for the defocusing cubic NLS on $\mathbb{T}^2$ [42, 51])**.** *Let u be the global solution of* (24) *on $\mathbb{T}^2$. Then, there exists a function $C = C_{s, \|u_0\|_{H^1}}$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C(1 + |t|)^{s+} \|u_0\|_{H^s(\mathbb{T}^2)}. \tag{25}$$

See also [8, 17].

Remark 4.2. *Let us note that, if we consider the spatial domain to be $\mathbb{R}^2$, one can obtain uniform bounds on $\|u(t)\|_{H^s}$ for solutions $u(t)$ of the defocusing cubic NLS by the recent scattering and highly non trivial result of Dodson* [23]*.*

The growth of high Sobolev norms has a physical interpretation in the context of the *low-to-high frequency cascade*. In other words, we see that $\|u(t)\|_{H^s}$ weighs the higher frequencies more as s becomes larger, and hence its growth gives us a quantitative estimate for how much of the support of $|\widehat{u}|^2$ has transferred from the low to the high frequencies. This sort of problem also goes under the name of *weak turbulence* [1, 49]. By local well-posedness theory discussed in Section 3, one can show that there exist $C, \tau_0 > 0$, depending only on the initial data u_0 such that for all t:

$$\|u(t + \tau_0)\|_{H^s} \leq C \|u(t)\|_{H^s}. \tag{26}$$

Iterating (26) yields the exponential bound:

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2 |t|}, \tag{27}$$

where $C_1, C_2 > 0$ again depend only on u_0.

For a wide class of nonlinear dispersive equations, the analogue of (27) can be improved to a polynomial bound, as long as we take $s \in \mathbb{N}$, or if we consider sufficiently smooth initial data. This observation was first made in the work of Bourgain [7], and was continued in the work of Staffilani [43, 44].

The crucial step in the mentioned works was to improve the iteration bound (26) to:

$$\|u(t+\tau_0)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-r}. \tag{28}$$

As before, $C, \tau_0 > 0$ depend only on u_0. In this bound, $r \in (0,1)$ satisfies $r \sim \frac{1}{s}$. One can show that (28) implies that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{\frac{1}{r}}. \tag{29}$$

In [7], (28) was obtained by using the *Fourier multiplier method*. In [43, 44], the iteration bound was obtained by using multilinear estimates in $X^{s,b}$-spaces due to Kenig-Ponce-Vega [31]. A slightly different approach, based on the analysis in the work of Burq-Gérard-Tzvetkov [11], is used to obtain (28) in the context of compact Riemannian manifolds in the work of Catoire-Wang [16], and Zhong [51].

The main idea in the proof of Theorem 4.1 in [42] is to introduce $\mathcal{D}$, an *upside-down I-operator*. This operator is defined as a Fourier multiplier operator. By construction, one is able to relate $\|u(t)\|_{H^s}$ to $\|\mathcal{D}u(t)\|_{L^2}$ and to consider the growth of the latter quantity. Following the ideas of the construction of the standard *I-operator*, as defined by Colliander, Keel, Staffilani, Takaoka, and Tao [18, 19, 20], the goal is to show that the quantity $\|\mathcal{D}u(t)\|_{L^2}^2$ is *slowly varying*. This is done by applying a Littlewood-Paley decomposition and summing an appropriate geometric series. A similar technique was applied in the low-regularity context in [19]. This first step though is not enough to prove Theorem 4.1. Instead one has to use *higher modified energies*, i.e. quantities obtained from $\|\mathcal{D}u(t)\|_{L^2}^2$ by adding an appropriate multilinear correction, again an idea introduced in [18, 19, 20]. In this way one obtains $E^2(u(t)) \sim \|\mathcal{D}u(t)\|_{L^2}^2$, which is even more slowly varying. Due to a complicated resonance phenomenon in two dimensions, the construction of E^2 is very involved and we do not present the details here.

4.1. **Example of energy transfer to high frequencies.** In this subsection we show that a very weak growth of Sobolev norms may indeed occur. More precisely we can prove

Theorem 4.3. *[Colliander-Keel-Staffilani-Takaoka-Tao,* [22]*] Let* $s > 1$, $K \gg 1$ *and* $0 < \sigma < 1$ *be given. Then there exist a global smooth solution* $u(x,t)$ *to the IVP* (24) *and* $T > 0$ *such that*

$$\|u_0\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T)\|_{\dot{H}^s}^2 \geq K.$$

We start by listing the elements of the proof. The first is a reduction to a resonant problem that we will refer to as the RFNLS system, see (32). Then in Subsection 4.2 we introduce a special finite set Λ of frequencies and we reduce the RFNLS system to a finite-dimensional Toy Model ODE system, see (33). We study this Toy Model dynamically in Subsection 4.3 and we show some sort of "sliding property" for it, see Theorem 4.4. In Subsection 4.4 we introduce the approximation Lemma 4.5 together with a scaling argument and finally in Subsection 4.5 we sketch the proof of Theorem 4.3.

We consider the gauge transformation

$$v(t,x) = e^{-i2Gt}u(t,x),$$

for $G \in \mathbb{R}$. If u solves the NLS (24) above, then v solves the equation

$$(-i\partial_t + \Delta)v = (2G+v)|v|^2.$$

We make the ansatz

$$v(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{i(\langle n,x\rangle + |n|^2 t)}.$$

Now the dynamics is all recast trough $a_n(t)$:

$$i\partial_t a_n = 2Ga_n + \sum_{n_1-n_2+n_3=n} a_{n_1}\overline{a_{n_2}}a_{n_3}e^{i\omega_4 t}, \tag{30}$$

where $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$. By choosing

$$G = -\|v(t)\|^2_{L^2} = -\sum_k |a_k(t)|^2$$

which is constant from the conservation of the mass, one can rewrite equation (30) as

$$i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a_{n_2}}a_{n_3}e^{i\omega_4 t}, \tag{31}$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 \;/\; n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n\}.$$

From now on we will be referring to this system as the $FNLS$ system, with the obvious connection with the original NLS equation.

We define the resonant set

$$\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) \,/\, \omega_4 = 0\}.$$

The geometric interpretation for this set is as follows: If n_1, n_2, n_3 are in $\Gamma_{res}(n)$, then the four points (n_1, n_2, n_3, n) represent the vertices of a rectangle in $\mathbb{Z}^2$. We finally define the resonant truncation $RFNLS$ to be the system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma_{res}(n)} b_{n_1}\bar{b}_{n_2}b_{n_3}. \tag{32}$$

We now would like to restrict the dynamics to a finite set of frequencies and this set would need several important properties. The first one is closeness under resonance. A finite set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \implies n = n_1 - n_2 + n_3 \in \Lambda.$$

Hence a Λ-finite dimensional resonant truncation of $RFNLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Gamma_{res}(n)\cap\Lambda^3} b_{n_1}\bar{b}_{n_2}b_{n_3}. \tag{33}$$

We will refer to this systems as the $RFNLS_\Lambda$ system.

4.2. **Λ: a very special set of frequencies.** We can construct a special Λ of frequencies with the following properties [22]

- **Generational set up:** $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$, N to be fixed later. A nuclear family is a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the 'parents') live in generation Λ_j and n_2, n_4 ('children') live in generation Λ_{j+1}.
- **Existence and uniqueness of spouse and children:** $\forall\ 1 \le j < N$ and $\forall\ n_1 \in \Lambda_j$ $\exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- **Existence and uniqueness of siblings and parents:** $\forall\ 1 \le j < N$ and $\forall\ n_2 \in \Lambda_{j+1}$ $\exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- **Non degeneracy:** The sibling of a frequency is never its spouse.
- **Faithfulness:** Besides nuclear families, Λ contains no other rectangles.

- **Intergenerational Equality:** The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.
- **Multiplicative Structure:** If $N = N(\sigma, K)$ is large enough then Λ consists of $N \times 2^{N-1}$ disjoint frequencies n with $|n| > R = R(\sigma, K)$, the first frequency in Λ_1 is of size R and we call R the *inner radius* of Λ. Moreover for any $n \in \Lambda$, $|n| \leq C(N)R$.
- **Wide Spreading:** Given $\sigma \ll 1$ and $K \gg 1$, if N is large enough then $\Lambda = \Lambda_1 \cup .... \cup \Lambda_N$ as above and

$$\sum_{n\in\Lambda_N} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n\in\Lambda_1} |n|^{2s}. \tag{34}$$

4.3. **The Toy Model.** The intergenerational equality hypothesis (that the function $n \longmapsto b_n(0)$ is constant on each generation Λ_j) persists under $RFNLS_\Lambda$ (33):

$$\forall\ m, n \in \Lambda_j,\ \ b_n(t) = b_m(t).$$

Also $RFNLS_\Lambda$ may be reindexed by generation index j and the recast dynamics is the Toy Model:

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)}, \tag{35}$$

with boundary condition

$$b_0(t) = b_{N+1}(t) = 0. \tag{36}$$

Using direct calculation[8], we will prove[9] that our Toy Model evolution $b_j(0) \longmapsto b_j(t)$ is such that:

$$\begin{aligned} (b_1(0), b_2(0), \ldots, b_N(0)) &\sim (1, 0, \ldots, 0) \\ (b_1(t_2), b_2(t_2), \ldots, b_N(t_2)) &\sim (0, 1, \ldots, 0) \\ &\cdot \\ (b_1(t_N), b_2(t_N), \ldots, b_N(t_N)) &\sim (0, 0, \ldots, 1) \end{aligned}$$

that is the bulk of conserved mass is transferred from Λ_1 to Λ_N and the weak transfer of energy from lower to higher frequencies follows from the *Wide Spreading* property (34) of Λ listed above.

We now make few observations that are simple, but they are nevertheless meant to show how nontrivial it is to move from Λ_1 to Λ_N. Global well-posedness for the Toy Model (35) is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^N \ / \ |x|^2 = 1\} \ \text{ and the flow map } \ W(t) : \Sigma \to \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of (35). It is easy to see that for any $b(t)$ with $b(0) \in \Sigma$

$$\partial_t |b_j|^2 = 4\mathrm{Re}(i\bar{b_j}^2(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 \implies b_j(t) = 0, \quad \text{for all} \quad t \in [0, T]$$

and if we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_N) \in \Sigma \,/\, |b_j| = 1,\, b_k = 0,\, k \neq j\}$$

then

$$W(t)\mathbb{T}_j = \mathbb{T}_j \ \text{ for all } j = 1, ..., N$$

[8]Maybe dynamical systems methods are useful here?

[9]See Theorem 4.4.

hence $\mathbb{T}_j$ is invariant. This suggests that if we want to move from a torus $\mathbb{T}_j$ to $\mathbb{T}_i$ we cannot start from data on $\mathbb{T}_j$ and moreover we need to show that we can manage to avoid to hit any $\mathbb{T}_k$, $j < k < i$. This is in fact the content of the following instability-type theorem:

Theorem 4.4. *[Sliding Theorem] Let $N \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{N-2} within ϵ of $\mathbb{T}_{N-2}$ and a time τ such that*

$$W(\tau)x_3 = x_{N-2}.$$

What the theorem says is that $W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time 0 and then gets moved so that it is concentrated near mode $j = N-2$ at later time τ.

For the complete, and unfortunately lengthy proof of this theorem see [22]. Here we only give a motivation for it that should clarify the dynamics involved. Let us first observe that when $N = 2$ we can easily demonstrate that there is an orbit connecting $\mathbb{T}_1$ to $\mathbb{T}_2$. Indeed in this case we have the explicit "slider" solution

$$b_1(t) := \frac{e^{-it}\omega}{\sqrt{1+e^{2\sqrt{3}t}}}; \quad b_2(t) := \frac{e^{-it}\omega^2}{\sqrt{1+e^{-2\sqrt{3}t}}} \tag{37}$$

where $\omega := e^{2\pi i/3}$ is a cube root of unity.

This solution approaches $\mathbb{T}_1$ exponentially fast as $t \to -\infty$, and approaches $\mathbb{T}_2$ exponentially fast as $t \to +\infty$. One can translate this solution in the j parameter, and obtain solutions that "slide" from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$. Intuitively, the proof of the Sliding Theorem for higher N should then proceed by concatenating these slider solutions....This though cannot work directly because each solution requires an infinite amount of time to connect one hoop to the next. It turned out though that a suitably perturbed or "fuzzy" version of these slider solutions can in fact be glued together.

4.4. **The Approximation lemma and the scaling argument.** There are still two steps we need to complete to prove Theorem 4.3. The first is to show that a solution to the Toy Model (35) is a good approximation for the solution to the original problem (31). This is accomplished with the following approximation lemma.

Lemma 4.5. *[Approximation Lemma] Let $\Lambda \subset \mathbb{Z}^2$ introduced above. Let $B \gg 1$ and $\delta > 0$ small and fixed. Let $t \in [0, T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Lambda)$ solving $RFNLS_\Lambda$ such that*

$$\|b(t)\|_{l^1} \lesssim B^{-1}. \tag{38}$$

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of $FNLS$ (31) such that for any $t \in [0, T]$

$$a(0) = b(0), \quad \text{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta}.$$

The proof for this lemma is pretty standard. The main idea is to check that the "non resonant" part of the nonlinearity remains small enough, see [22] for details.

The last ingredient before we proceed to the proof of our main result is *the scaling argument*. What we proved so far is that we can find a solution of mass one that a time zero is localized in Λ_3 and if we wait long enough will be localized in Λ_{N-2}. But what Theorem 4.3 asks is a solution that is "small" at time zero. This is why we need to introduce scaling. It is easy to check that if $b(t)$ solves $RFNLS_\Lambda$ (33) then the rescaled solution

$$b^\lambda(t) = \lambda^{-1} b(\frac{t}{\lambda^2})$$

solves the same system with datum $b_0^\lambda = \lambda^{-1} b_0$.

We then pick the complex vector $b(0)$ that was found in the Sliding Theorem 4.4 above. For simplicity let us assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} b_j^\lambda(0) & \text{for any } \ n \in \Lambda_j \\ 0 & \text{otherwise} . \end{cases} \tag{39}$$

We are now ready to finish the proof of Theorem 4.3. For simplicity we recast it with all the notations and reductions introduced so far:

Theorem 4.6. *For any* $0 < \sigma \ll 1$ *and* $K \gg 1$ *there exists a complex sequence* (a_n) *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ *of* $FNLS$ *and* $T > 0$ *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$

4.5. **Proof of Theorem 4.6.** We start by estimating the size of $(a_n(0))$. By definition

$$\left(\sum_{n \in \Lambda} |a_n(0)|^2 |n|^{2s} \right)^{1/2} = \frac{1}{\lambda} \left(\sum_{j=1}^{M} |b_j(0)|^2 \left(\sum_{n \in \Lambda_j} |n|^{2s} \right) \right)^{1/2} \sim \frac{1}{\lambda} Q_3^{1/2},$$

where the last equality follows from defining

$$\sum_{n \in \Lambda_j} |n|^{2s} = Q_j,$$

and the definition of $a_n(0)$ given in (39). At this point we use the proprieties of the set Λ to estimate $Q_3 \sim C(N) R^{2s}$, where R is the inner radius of Λ. We then conclude that

$$\left(\sum_{n \in \Lambda} |a_n(0)|^2 |n|^{2s} \right)^{1/2} = \lambda^{-1} C(N) R^s \sim \sigma,$$

for a large enough R.

Now we want to estimate the size of $(a_n(T))$. Take $B = \lambda$ and $T = \lambda^2 \tau$ in Lemma 4.5 and write

$$\|a(T)\|_{H^s} \geq \|b^\lambda(T)\|_{H^s} - \|a(T) - b^\lambda(T)\|_{H^s} = I_1 - I_2.$$

We want $I_2 \ll 1$ and $I_1 > K$. For I_2 we use the Approximation Lemma 4.5

$$I_2 \lesssim \lambda^{-1-\delta} \left(\sum_{n \in \Lambda} |n|^{2s} \right)^{1/2} \lesssim \lambda^{-1-\delta} C(N) R^s.$$

At this point we need to pick λ and N so that

$$\|a(0)\|_{H^s} = \lambda^{-1} C(N) R^s \sim \sigma \quad \text{and} \quad I_2 \lesssim \lambda^{-1-\delta} C(N) R^s \ll 1$$

and thanks to the presence of $\delta > 0$ this can be achieved by taking λ and R large enough.

Finally we estimating I_1. It is important here that at time zero one starts with a fixed non zero datum, namely $\|a(0)\|_{H^s} = \|b^\lambda(0)\|_{H^s} \sim \sigma > 0$. In fact we will show that

$$I_1^2 = \|b^\lambda(T)\|_{H^s}^2 \geq \frac{K^2}{\sigma^2}\|b^\lambda(0)\|_{H^s}^2 \sim K^2.$$

If we define for $T = \lambda^2 t$

$$R = \frac{\sum_{n\in\Lambda} |b_n^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{n\in\Lambda} |b_n^\lambda(0)|^2 |n|^{2s}},$$

then we are reduce to showing that $R \gtrsim K^2/\sigma^2$. Recall the notation

$$\Lambda = \Lambda_1 \cup ..... \cup \Lambda_N \quad \text{and} \quad \sum_{n\in\Lambda_j} |n|^{2s} = Q_j.$$

Using the fact that by the Sliding Theorem 4.4 one obtains $b_j(T) = 1-\epsilon$ if $j = N-2$ and $b_j(T) = \epsilon$ if $j \neq N-2$, it follows that

$$\begin{aligned} R &= \frac{\sum_{i=1}^N \sum_{n\in\Lambda_i} |b_i^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^N \sum_{n\in\Lambda_i} |b_i^\lambda(0)|^2 |n|^{2s}} \\ &\geq \frac{Q_{N-2}(1-\epsilon)}{(1-\epsilon)Q_3 + \epsilon Q_1 + ... + \epsilon Q_N} \sim \frac{Q_{N-2}(1-\epsilon)}{Q_{N-2}\left[(1-\epsilon)\frac{Q_3}{Q_{N-2}} + ... + \epsilon\right]} \\ &\gtrsim \frac{(1-\epsilon)}{(1-\epsilon)\frac{Q_3}{Q_{N-2}}} = \frac{Q_{N-2}}{Q_3} \end{aligned}$$

and the conclusion follows from the "Wide Spreading" property (34) of Λ_j:

$$Q_{N-2} = \sum_{n\in\Lambda_{N-2}} |n|^{2s} \gtrsim \frac{K^2}{\sigma^2} \sum_{n\in\Lambda_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3.$$

5. Periodic Schrödinger equations as infinite dimension Hamiltonian systems

In this section we are going to view some Schrödinger equations as infinite dimension Hamiltonian systems. We will show two results generalizing to the infinite dimensional setting two important concepts such as the invariance of the Gibbs measure and the non-squeezing lemma of Gromov [24]. In the next subsection we recall these two concepts in more details.

5.1. **The finite dimension case.** Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i} \tag{40}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}(-\frac{\partial H}{\partial q_i}) = 0.$$

By defining $y := (q_1, \dots, q_k, p_1, \dots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite (40) in the compact form

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We now recall the following theorem giving a sufficient condition under which a flow map preserves the volume:

Theorem 5.1. *[Liouville's Theorem] Let a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ be divergence free. If the flow map Φ_t satisfies*

$$\frac{d}{dt}\Phi_t(y) \;=\; f(\Phi_t(y)),$$

then Φ_t is a volume preserving map for all t.

In particular if f is associated to a Hamiltonian system then automatically $div\, f = 0$. As a consequence the Lebesgue measure on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow of (40).

There are other measures that remain invariant[10] under the Hamiltonian flow: the Gibbs measures. In fact we have

Theorem 5.2. *[Invariance of Gibbs measures] Assume that Φ_t is the flow generated by the Hamiltonian system (40). Then the Gibbs measures*

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i\, dq_i$$

with $\beta > 0$, are invariant under the flow Φ_t.

The proof is trivial since from conservation of the Hamiltonian H the functions $e^{-\beta H(p,q)}$ remain constant, while, thanks to the Liouville's Theorem 5.1 the volume $\prod_{i=1}^{d} dp_i\, dq_i$ remains invariant as well.

Next result, much more difficult to prove, is the non-squeezing theorem:

Theorem 5.3 (Non-squeezing [24])**.** *Assume that Φ_t is the flow generated by the Hamiltonian system (40). Fix $y_0 \in \mathbb{R}^{2k}$ and let $B_r(y_0)$ be the ball in $\mathbb{R}^{2k}$ centered at y_0 and radius r. If $C_R(z_0) := \{y = (q_1, \dots, q_k, p_1, \dots, p_k) \in \mathbb{R}^{2k} / |q_i - z_0| \le R\}$, a cylinder of radius R, and $\Phi_t(B_r(y_0)) \subset C_R(z_0)$, it must be that $r \le R$.*

We now would like to see if Theorem 5.2 and Theorem 5.3 can be generalized to an infinite dimensional setting.

5.2. **Periodic Schrödinger equations and Gibbs measures.** Let us go back to (1). One can use $H(u, \bar{u})$ and check that equation (1) is equivalent to

$$\dot{u} = i\frac{\partial H(u, \bar{u})}{\partial \bar{u}}$$

where $H(t)$ is the Hamiltonian defined in (2), and one can think of u as the infinite dimension vector given by its Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}^n} = (a_k, b_k)_{k\in\mathbb{Z}^n}$.

Lebowitz, Rose and Speer [34] considered the Gibbs measure *formally* given by

$$\text{“}d\mu = Z^{-1} \exp\left(-\beta H(u)\right) \prod_{x\in\mathbb{T}} du(x)\text{”} \tag{41}$$

and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$, see Remark 5.6.

[10] A measure μ remains invariant under a flow Φ_t if for any A, subset of the support of μ, one has

$$\mu(\Phi_t(A)) = \mu(A).$$

Bourgain [5] proved the invariance of this measure and almost surely global well-posedness of the associated initial value problem[11]. For example, for $p = 4$ in (1) he proved:

Theorem 5.4. *Consider the NLS initial value problem*

$$\begin{cases} (i\partial_t + \Delta)u = \lambda |u|^4 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}. \end{cases} \tag{42}$$

If $\lambda = 1$ (defocusing case) the measure μ (41) is well defined in H^s, $0 < s < 1/2$ and there exists $\Omega \subset H^s$ such that $\mu(\Omega) = 1$ and (42) is globally well-posed in Ω. Moreover the measure μ is invariant under the flow given by (42). If $\lambda = -1$ (focusing case), then a similar result holds for

$$d\mu = Z^{-1} \chi_{\{\|u\|_L^2 \leq B\}} \exp\left(-\beta H(u)\right) \prod_{x \in \mathbb{T}} du(x)$$

with B small enough.

Remark 5.5. *If one considers the IVP* (1) *in the focusing case, then Theorem 5.4 only holds for $p \leq 5$, but if $p < 5$ we can take $B > 0$ arbitrary, see* [5].

After Bourgain's result recalled above, almost surely global well-posedness for a variety of IVP has been studied by introducing invariant measures. See for example Burq and Tzevtkov for subcubic and subquartic radial NLW on 3D balls [12, 13], T. Oh for the periodic KdV-type and Schrödinger-Benjamin-Ono coupled systems [35, 36, 37], Oh-Nahmod-Rey-Bellet-Staffilani [38] and Thomann and Tzevtkov [48] for the periodic derivative NLS equation. This last one will be the subject of Subsection 5.5.

5.3. **Gaussian measures and Gibbs measures.** In this subsection I would like to elaborate a little more on Gaussian and Gibbs measures by using as an example the measure that is naturally attached to the IVP (42) above. Note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is conserved. Then the best way to make sense of the Gibbs measure μ formally defined in (41) is by writing it as

$$d\mu = Z^{-1}\chi_{\|u\|_{L^2} \leq B} \exp\left(\frac{1}{6}\int |u|^6\,dx\right) \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right) \prod_{x \in \mathbb{T}} du(x).$$

In this expression

$$d\rho = \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right) \prod_{x \in \mathbb{T}} du(x)$$

is the Gaussian measure and

$$\frac{d\mu}{d\rho} = \chi_{\|u\|_{L^2} \leq B} \exp\left(\frac{1}{6}\int |u|^6\,dx\right),$$

corresponding to the nonlinear term of the Hamiltonian, is understood as the Radon-Nikodym derivative of μ with respect to ρ.

[11] The remarkable fact is that this statement is true both in the focusing and defocusing case, modulo of course the restriction on the L^2 norm in Remark 5.5.

The Gaussian measure ρ is defined as the weak limit of the finite dimensional Gaussian measures

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\le N} (1+|n|^2)|\widehat{u}_n|^2\Big) \prod_{|n|\le N} da_n db_n \,.$$

For a precise definition of Gaussian measures on Hilbert and Banach spaces in general see [25, 33]. Here we briefly recall how one shows that Sobolev spaces $H^s(\mathbb{T})$ are supports for ρ only if $s<\frac{1}{2}$. Consider the operator $\mathcal{J}_s = (1-\Delta)^{s-1}$. Then

$$\sum_n (1+|n|^2)\,|\widehat{u}_n|^2 \;=\; \langle u,u\rangle_{H^1} \;=\; \langle \mathcal{J}_s^{-1} u, u\rangle_{H^s} \,.$$

The operator $\mathcal{J}_s : H^s \to H^s$ has the set of eigenvalues $\{(1+|n|^2)^{(s-1)}\}_{n\in\mathbb{Z}}$ and the corresponding eigenvectors $\{(1+|n|^2)^{-s/2}e^{inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of H^s. For ρ to be *countable additive* we need $\mathcal{J}_s$ to be of *trace class* which is true if and only if $s<\frac{1}{2}$. Then ρ is a countably additive measure on H^s for any $s<\frac{1}{2}$. See again [25, 33].

The following remark is meant to explain the probabilistic aspect of Gibbs measures.

Remark 5.6. *The measure ρ_N above can be regarded as the induced probability measure on $\mathbb{R}^{4N+2}$ under the map*

$$\omega \longmapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{|n|\le N} \qquad \textit{and} \qquad \widehat{u}_n = \frac{g_n}{\sqrt{1+|n|^2}},$$

where $\{g_n(\omega)\}_{|n|\le N}$ are independent standard complex Gaussian random variables on a probability space $(\Omega,\mathcal{F},P)$.

In a similar manner, we can view ρ as the induced probability measure under the map

$$\omega \mapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{n\in\mathbb{Z}} .$$

5.4. **Bourgain's Method.** Above we stated Bourgain's theorem for the quintic focusing periodic NLS. Here we give an outline of Bourgain's idea in a general framework, and discuss how to prove almost surely global well-posedness and the invariance of a measure starting with a local well-posedness result.

Consider a dispersive nonlinear Hamiltonian PDE with a k-linear nonlinearity, possibly with derivative:

$$\begin{cases} u_t = \mathcal{L}u + \mathcal{N}(u) \\ u|_{t=0} = u_0, \end{cases} \tag{43}$$

where $\mathcal{L}$ is a (spatial) differential operator like $i\partial_{xx}$, ∂_{xxx}, etc. Let $H(u)$ denote the Hamiltonian of (43). Then (43) can also be written as

$$u_t = J\,\frac{dH}{du} \quad \text{ if } u \text{ is real-valued,} \quad u_t = J\,\frac{\partial H}{\partial \overline{u}} \quad \text{ if } u \text{ is complex-valued,}$$

for an appropriate operator J. Let μ denote a measure on the distributions on $\mathbb{T}$, whose invariance we would like to establish. We assume that μ is (formally) given by

$$\text{“}\, d\mu = Z^{-1} e^{-F(u)} \prod_{x\in\mathbb{T}} du(x)\,\text{”},$$

where $F(u)$ is conserved[12] under the flow of (43) and the leading term of $F(u)$ is quadratic and nonnegative. Now, suppose that there is a good local well-posedness theory, that is there exists a Banach space $\mathcal{B}$ of distributions on $\mathbb{T}$ and a space $X_\delta \subset C([-\delta,\delta];\mathcal{B})$ of space-time distributions in which one proves local well-posedness by a fixed point argument with a time of existence δ depending on $\|u_0\|_B$, say $\delta \sim \|u_0\|_B^{-\alpha}$ for some $\alpha > 0$. In addition, suppose that the Dirichlet projections P_N – the projection onto the spatial frequencies $\leq N$ – act boundedly on these spaces, uniformly in N. Then for $\|u_0\|_B \leq K$ the finite dimensional approximation to (43)

$$\begin{cases} u_t^N = \mathcal{L}u^N + P_N\big(\mathcal{N}(u^N)\big) \\ u^N|_{t=0} = u_0^N := P_N u_0(x) = \sum_{|n|\leq N} \widehat{u}_0(n)e^{inx}. \end{cases} \tag{44}$$

is locally well-posed on $[-\delta,\delta]$ with $\delta \sim K^{-\alpha}$, independent of N. We need two more important assumptions on (44): that (44) is Hamiltonian with $H(u^N)$ i.e.

$$u_t^N = J\,\frac{dH(u^N)}{d\overline{u}^N} \tag{45}$$

and that

$$\frac{d}{dt}F(u^N(t)) = 0, \tag{46}$$

that is $F(u^N)$ is still conserved under the flow of (44).

Note that the first holds for example when J commutes with the projection P_N, (e.g. $J = i$ or ∂_x.). In general however the two assumptions above are not guaranteed and may not necessarily hold. See Subsection 5.5.

At this point we have:

- By Liouville's theorem and (45) the Lebesgue measure $\prod_{|n|\leq N} da_n db_n$, where $\widehat{u^N}(n) = a_n + ib_n$, is invariant under the flow of (44).
- Using (46) - the conservation of $F(u^N)$- the finite dimensional version μ_N of μ:
$$d\mu_N = Z_N^{-1} e^{-F(u^N)} \prod_{|n|\leq N} da_n db_n$$
is also invariant under the flow of (44).

The next ingredient we need is:

Lemma 5.7 (Fernique-type tail estimate)**.** *For K sufficiently large, we have*

$$\mu_N\big(\{\|u_0^N\|_{\mathcal{B}} > K\}\big) < Ce^{-CK^2},$$

where all constants are independent of N.

This lemma and the invariance of μ_N imply the following estimate controlling the growth of the solution u^N to (44) [5].

Proposition 5.8. *Given $T < \infty$, $\varepsilon > 0$, there exists $\Omega_N \subset \mathcal{B}$ such that $\mu_N(\Omega_N^c) < \varepsilon$ and for $u_0^N \in \Omega_N$, (44) is well-posed on $[-T,T]$ with the growth estimate:*

$$\|u^N(t)\|_{\mathcal{B}} \lesssim \Big(\log\frac{T}{\varepsilon}\Big)^{\frac{1}{2}}, \ \textit{for } |t| \leq T.$$

[12] $F(u)$ could be the Hamiltonian, but not necessarily!

Proof. Let $\Phi_N(t)$ be the flow map of (44), and define

$$\Omega_N = \cap_{j=-[T/\delta]}^{[T/\delta]} \Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}} \leq K\}).$$

By invariance of μ_N,

$$\mu(\Omega_N^c) = \sum_{j=-[T/\delta]}^{[T/\delta]} \mu_N \left(\Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}} > K\}\right) = 2[T/\delta]\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\})$$

This implies $\mu(\Omega_N^c) \lesssim \frac{T}{\delta}\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\}) \sim TK^\theta e^{-cK^2}$, and by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}$, we have $\mu(\Omega_N^c) < \varepsilon$. By its construction, $\|u^N(j\delta)\|_{\mathcal{B}} \leq K$ for $j = 0, \cdots, \pm[T/\delta]$ and by local theory,

$$\|u^N(t)\|_{\mathcal{B}} \leq 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} \text{ for } |t| \leq T.$$

□

One then needs to prove that μ_N converges weakly to μ. This is standard and one can check the work of Zhidkov [50] for example. Going back to (43), essentially as a corollary of Proposition 5.8 one can then prove:

Corollary 5.9. *Given $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \mathcal{B}$ with $\mu(\Omega_\varepsilon^c) < \varepsilon$ such that for $u_0 \in \Omega_\varepsilon$, the IVP* (43) *is globally well-posed and*
(a) $\|u - u^N\|_{C([-T,T];\mathcal{B}')} \to 0$ as $N \to \infty$ uniformly for $u_0 \in \Omega_\varepsilon$, where $\mathcal{B}' \supset \mathcal{B}$.
(b) One has the growth estimate

$$\|u(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{1+|t|}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for all } t \in \mathbb{R}.$$

One can prove (a) and (b) by estimating the difference $u - u^N$ using the local well-posedness theory and a standard approximation lemma, and then applying Proposition 5.8 to u^N. Finally if $\Omega := \bigcup_{\varepsilon>0} \Omega_\varepsilon$, clearly $\mu(\Omega) = 1$ and (43) is almost surely globally well-posed. At the same time one also obtains the invariance of μ.

5.5. **The periodic, one dimensional derivative Schrödinger equation.** It is now time to introduce another infinite dimensional system: the derivative NLS equation (DNLS)

$$\begin{cases} u_t - i\, u_{xx} = \lambda(|u|^2 u)_x, \\ u\big|_{t=0} = u_0, \end{cases} \tag{47}$$

where $(x,t) \in \mathbb{T} \times (-T, T)$ and λ is real. Below we will take $\lambda = 1$ for convenience. We note that DNLS is an Hamiltonian PDE. In fact, it is completely integrable [28]. The first three conserved integrals are:

$$\text{Mass:} \quad m(u) = \frac{1}{2\pi} \int_{\mathbb{T}} |u(x,t)|^2\, dx$$

$$\text{`Energy':} \quad E(u) = \int_{\mathbb{T}} |u_x|^2\, dx + \frac{3}{2}\text{Im} \int_{\mathbb{T}} u^2 \overline{u u_x}\, dx + \frac{1}{2}\int_{\mathbb{T}} |u|^6\, dx =: \int_{\mathbb{T}} |u_x|^2\, dx + \mathcal{K}(u) \tag{48}$$

$$\text{Hamiltonian:} \quad H(u) = \text{Im} \int_{\mathbb{T}} u\overline{u}_x\, dx + \frac{1}{2}\int_{\mathbb{T}} |u|^4\, dx.$$

We would like now to explore the possibility of extending Bourgain's approach to the periodic DNLS (47). We should immediately say that Thomann and Tzvetkov [48] already proposed a measure for this problem. Unfortunately though the presence of the derivative term in the nonlinearity, in particular $|u|^2u_x$, makes it impossible to prove the needed multilinear estimates of the type presented in Section 3, that are the fundamental tools to show both invariance and almost surely global well-posedness. For this reason one needs to remove the term $|u|^2u_x$ by gauging via the transformation [28, 46]

$$G(f)(x) := \exp(-iJ(f))\, f(x) \tag{49}$$

where

$$J(f)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})} \right) dy\, d\theta \tag{50}$$

is the unique 2π-periodic mean zero primitive of the map

$$x \longrightarrow |f(x)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}.$$

Then, for $u \in C([-T,T]; L^2(\mathbb{T}))$ the adapted periodic gauge is defined as

$$\mathcal{G}(u)(t,x) := G(u(t))(x - 2\,t\,m(u)).$$

Note that the difference between $\mathcal{G}$ and G is a space translation by $2\,t\,m(u)$ and this is introduced simply to remove an extra linear term that would have appeared in the gauged equation if one had only used G. We have that

$$\mathcal{G} : C([-T,T]; H^\sigma(\mathbb{T})) \to C([-T,T]; H^\sigma(\mathbb{T}))$$

is a homeomorphism for any $\sigma \geq 0$. Moreover, $\mathcal{G}$ is locally bi-Lipschitz on subsets of functions in $C([-T,T]; H^\sigma(\mathbb{T}))$ with prescribed L^2-norm. The same is true if we replace $H^\sigma(\mathbb{T})$ by $\mathcal{F}L^{s,r}$, the Fourier-Lebesgue spaces defined in (55) below. If u is a solution to (47) then $v := \mathcal{G}(u)$ is a solution to the gauged DNLS initial value problem, here denoted GDNLS:

$$v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v \tag{51}$$

with initial data $v(0) = \mathcal{G}(u(0))$, where

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v\overline{v}_x)\, dx \, + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4 dx - m(v)^2$$

and

$$m(u) = m(v) := \frac{1}{2\pi}\int_{\mathbb{T}} |v|^2(x,t)dx = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2(x)dx$$

is the conserved mass. One can also check that if

$$\begin{aligned} \mathcal{E}(v) &:= \int_{\mathbb{T}} |v_x|^2\, dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v\,v_x}\, dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v(t)|^2\, dx\right)\left(\int_{\mathbb{T}} |v(t)|^4\, dx\right), \\ \mathcal{H}(v) &:= \mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x - \frac{1}{2}\int_{\mathbb{T}} |v|^4\, dx + 2\pi m(v)^2 \end{aligned}$$

and

$$\tilde{\mathcal{E}}(v) := \mathcal{E}(v) + 2m(v)\mathcal{H}(v) - 2\pi\, m(v)^3 \tag{52}$$

then all are conserved integrals. For convenience let us write

$$\tilde{\mathcal{E}}(v) = \int_{\mathbb{T}} |v_x|^2\, dx + \mathcal{N}(v), \tag{53}$$

where $\mathcal{N}(v)$ represents the part of the energy that comes from the nonlinearity. We now define, at least formally the measure μ as

$$\text{“}d\mu = Z^{-1}\chi_{\{\|v\|_{L^2}<B\}}e^{\mathcal{N}(v)}d\rho\text{”}, \tag{54}$$

where the cut-off function with respect to the L^2 norm is suggested by Remark 5.5 and the fact that equation (51) has a quintic term in it. The plan is then is to show that for B small this measure is well defined and invariant for the GDNLS, that GDNLS is almost surely global well-posedness with respect to it and finally that one can un-gauge to go back to the DNLS (47). Unfortunately there are several obstacles that one needs to overcome to implement this plan. The first is that (51) is ill posed[13] in $H^s, s < \frac{1}{2}$, see [3]. On the other hand Grünrock-Herr [27] proved local well-posedness for initial data $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ and $2 \leq r < 4$, $s \geq \frac{1}{2}$, where

$$\|v_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \| \langle n \rangle^s \hat{v}_0 \|_{\ell^r_n(\mathbb{Z})} \qquad r \geq 2, \tag{55}$$

avoiding in this way L^2 based Sobolev spaces. These spaces scale like the Sobolev spaces $H^\sigma(\mathbb{T})$, where $\sigma = s + 1/r - 1/2$. For example for $s = \frac{2}{3}-$ and $r = 3$ one has that $\sigma < \frac{1}{2}$. As a result one can use Gaussian measures on Banach spaces [14] $\mathcal{F}L^{s,r}(\mathbb{T})$. The next issue is the fact that when one projects (51) via P_N the resulting IVP

$$v_t^N = iv_{xx}^N - P_N((v^N)^2\overline{v_x^N}) + \frac{i}{2}P_N(|v^N|^4v^N) - i\psi(v^N)v^N - im(v^N)P_N(|v^N|^2v^N) \tag{56}$$

with initial data $v_0^N = P_N\, v_0$ is no longer in an Hamiltonian form that one can recognize and one needs to prove Liouville's theorem by hands. The final, and probably the most serious problem is that the energy $\tilde{\mathcal{E}}(v)$ in (53), that is conserved for (51), is no longer conserved when one projects via P_N. Fortunately though Bourgain's argument can be made more general, in particular it is enough to show that $\tilde{\mathcal{E}}(v^N)$ is *almost* conserved. At the end one can show

Theorem 5.10. *[Almost sure global well-posedness of GDNLS* (51) *and invariance] The measure μ in* (54) *is well defined on $\mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})$. Moreover there exists $\Omega \subset \mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})$, $\mu(\Omega^c) = 0$ such that GDNLS* (51) *is globally well-posed in Ω and μ is invariant on Ω.*

The last step, a pretty straightforward one, is going back to the un-gauged equation DNLS (47) by pulling back the gauge, that is by defining $\nu := \mu\circ\mathcal{G}$ and in so doing obtaining a theorem like Theorem 5.10 for the initial value problem DNLS (47) and the measure ν, see [38] for details. Now the interesting question is to understand what $\nu = \mu \circ \mathcal{G}$ really represents. Is ν absolutely continuous with respect to the measure that can be naturally constructed for DNLS by using its energy E in (48), as done by Thomann-Tzevtkov [48]?

When we un-gauge the measure μ, at least formally we are un-gauging two pieces, the Radon-Nikodym derivative and the Gaussian measure. Treating the Radon-Nikodym derivative is easy. The problem is un-gauging the Gaussian measure ρ. We can ask the following question: What is $\tilde{\rho} := \rho \circ \mathcal{G}$? Is (its restriction to a sufficiently small ball in L^2) absolutely continuous with respect to ρ? If so, what is its Radon-Nikodym derivative?

[13] The ill-posedness result has actually been proved only in $\mathbb{R}$ so far and it says that a fixed point argument cannot be used in Sobolev spaces based L^2. It is believed that this negative result is also true in the periodic case.

[14] For this reason at the end of the day one will be talking about weighted Wiener measures instead of Gibbs measures, see [38] for more details.

5.6. **Gaussian measures and gauge transformations.** In order to finish this step one should stop thinking about the solution v as a infinite dimension vector of Fourier modes and instead start thinking about v as a (periodic with period 1) complex Brownian path in $\mathbb{T}$ (Brownian bridge) solving a certain stochastic process. The argument that follows can be found in full details in [40].

We notice from (49) that to un-gauge we need to use

$$\mathcal{G}^{-1}(v)(x) = \exp(iJ(v))\, v(x)$$

where $J(v)(x)$ was defined in (50). It will be important later that $J(v)(x) = J(|v|)(x)$. Then, if v satisfies

$$dv(x) = \underbrace{dB(x)}_{\text{Brownian motion}} + \underbrace{b(x)dx}_{\text{drift terms}}$$

by Ito's calculus and since $\exp(iJ(v))$ is differentiable we have:

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\, dv + iv\exp(iJ(v))\left(|v(x)|^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right) dx + \dots.$$

Substituting above one has

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\,[dB(x) + a(v,x,\omega))\,dx] + \dots$$

where

$$a(v,x,\omega) = iv\ \left(|v(x)|^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right). \tag{57}$$

What could help? Certainly the fact that $\exp(iJ(v))$ is a unitary operator and that one can prove Novikov's condition:

$$E\left[\exp\left(\frac{1}{2}\int a^2(v,x,\omega)dx\right)\right] < \infty. \tag{58}$$

In fact this last condition looks exactly like what we need for the following theorem:

Theorem 5.11 (Girsanov [39]). *If we change the drift coefficient of a given Ito process in an appropriate way, see* (57)*, then the law of the process will not change dramatically. In fact the new process law will be absolutely continuous with respect to the law of the original process and we can compute explicitly the Radon-Nikodym derivative.*

Unfortunately though Girsanov's theorem doesn't save the day.... at least not immediately. If one reads the theorem carefully one realizes that an important condition is that $a(v,x,\omega)$ is *non anticipative*, in the sense that it only depends on the BM v up to "time" x and not further. This unfortunately is not true in our case. The new drift term $a(v,x,\omega)$ involves the L^2 norm of $v(x)$, see (57), and hence it is *anticipative*. A different strategy is needed and conformal invariance of complex BM comes to the rescue

We use the well known fact that if $W(t) = W_1(t)+iW_2(t)$ is a complex Brownian motion, and if ϕ is an analytic function then $Z = \phi(W)$ is, after a suitable time change, again a complex Brownian motion[15], [39]. For $Z(t) = \exp(W(s))$ the time change is given by

$$t = t(s) = \int_0^s |e^{W(r)}|^2 dr, \qquad s(t) = \int_0^t \frac{dr}{|Z(r)|^2}.$$

[15] In what follows one should think of $Z(t)$ to play the role of our complex BM $v(x)$.

We are interested on $Z(t)$ for the interval $0 \leq t \leq 1$ and thus we introduce the stopping time

$$\mathcal{S} = \inf \left\{ s\,;\, \int_0^s |e^{W(r)}|^2 dr = 1 \right\}$$

and remark the important fact that the stopping time $\mathcal{S}$ depends only on the real part $W_1(s)$ of $W(s)$ (or equivalently only $|Z|$). If we write $Z(t)$ in polar coordinate $Z(t) = |Z(t)|e^{i\Theta(t)}$, we have

$$W(s) = W_1(s) + iW_2(s) = \log|Z(t(s)| + i\Theta(t(s))$$

and W_1 and W_2 are real independent Brownian motions. If we define

$$\tilde{W}(s) := W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(|Z|)(r)dr\right] = W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(e^{W_1})(r)dr,\right]$$

then have

$$e^{\tilde{W}(s)} = \tilde{Z}(t(s)) = \mathcal{G}^{-1}(Z)(t(s)).$$

In terms of W, the gauge transformation is now easy to understand: it gives a complex process in which the real part is left unchanged and the imaginary part is translated by the function $J(Z)(t(s))$ in (50) which depends only on the real part (i.e. on $|Z|$, which has been fixed) and in that sense is deterministic. It is now possible to use the Cameron-Martin-Girsanov's Theorem [14, 39] only for the law of the imaginary part and conclude the proof. Then if η denotes the probability distribution of W and $\tilde{\eta}$ the distribution of $\tilde{W}$ we have the absolute continuity of $\tilde{\eta}$ and η whence the absolute continuity between $\tilde{\rho}$ and ρ follows with the *same Radon-Nikodym derivative* (re-expressed back in terms of t). All in all then we prove that our un-gauged measure ν is in fact essentially (up to normalizing constants) of the form

$$d\nu(u) = \chi_{\|u\|_{L^2} \leq B} e^{-\mathcal{K}(u)} d\rho,$$

where $\mathcal{K}(u)$ was introduced in (48), that is the weighted Wiener measure associated to DNLS (constructed by Thomann-Tzvetkov [48]). In particular we prove its invariance.

Remark 5.12. *The sketch of the argument above needs to be done carefully for complex Brownian bridges (periodic BM) by conditioning properly. See* [40].

5.7. **Periodic dispersive equations and the non-squeezing theorem.** In Theorem 5.3 we recalled like a finite dimensional Hamiltonian flow Φ_t cannot squeeze a ball into a cylinder with a smaller radius. Generalizing this kind of result in infinite dimensions has been a long project of Kuksin [32] who proved, roughly speaking, that compact perturbations of certain linear dispersive equations do indeed satisfy the non-squeezing theorem. It is easy to show that the L^2 space equipped with the form

$$\omega(f, g) = \langle if, g\rangle_{L^2} \tag{59}$$

is a symplectic space for the cubic, defocusing NLS equation on $\mathbb{T}$ and its global flow $\Phi(t)$ is a symplectomorphism. One can show that this setting does not satisfy the conditions in [32]. Nevertheless Bourgain proved the following theorem:

Theorem 5.13. *[Non-squeezing* [5]*] Assume that Φ_t is the flow generated by the cubic, periodic, defocusing NLS equation in L^2. If we identify L^2 with l^2 via Fourier transform and we let $B_r(y_0)$ be the ball in l^2 centered at $y_0 \in l^2$ and radius r, $C_R(z_0) := \{(a_n) \in l^2 / |a_i - z_0| \leq R\}$ a cylinder of radius R and $\Phi_t(B_r(y_0)) \subset C_R(z_0)$, at some time t, then it must be that $r \leq R$.*

The proof of this theorem is based on projecting the IVP onto finitely many frequencies via the projection operator P_N as was done in (44). In this case the new projected problem is a finite dimensional Hamiltonian system and Gromov's Theorem 5.3 can be applied. The difficult part then is to show that the flow $\Phi_N(t)$ of the projected IVP approximates well the flow $\Phi(t)$ of the original problem. In this case this can be done thanks to strong multilinear estimates based on the Strichartz estimates recalled in Theorem 18; see [5] for the complete proof. We should mention here that unfortunately Bourgain's argument may not work for other kinds of dispersive equations. For example in [21], where the KdV problem was studied, the lemma in Bourgain's work that gives the good approximation of the flow $\Phi(t)$ by $\Phi_N(t)$ does not hold. This has to do with the number of interacting waves in the nonlinearity. There it was proved that still the non-squeezing theorem holds, but the proof was indirect and it had to go through the Miura transformation.

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