Abstract

We determine the diameter of the 1-skeleton and the combinatorial automorphism group of any Gelfand-Tsetlin polytope $GT_{\lambda}$ associated to an integer partition $\lambda$.

Introduction

Gelfand-Tsetlin (GT) polytopes are compact convex polytopes defined by a set of linear inequalities depending on a partition $\lambda$ as shown in Figure 1. The polytope $GT_{\lambda}$ corresponds to the set of points $\mathcal{F} = (x_{i,j})_{1 \leq i \leq j \leq \lambda} \in \mathbb{R}^{n(n+1)/2}$ where $(x_{i,j})_{1 \leq i \leq j \leq \lambda}$ is a filling of this triangular array such that all rows and columns are weakly increasing.

Example: $GT_{\lambda, \lambda} = (1, 2, 3)$

One can see from this figure that the diameter of the 1-skeleton is 2 and there are 4 automorphisms.

Background

$GT_{\lambda}$ polytopes arise from the study of representations of $GL_n(\mathbb{C})$ and have connections to areas of representation theory and algebraic geometry. For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, let $n$ be the length of $\lambda$ and let $GT_{\lambda}$ denote the associated GT polytope. The integral points in $GT_{\lambda}$ are in bijection with semi-standard Young tableaux of shape $\lambda$ with tableaux entries bounded by $n$. Furthermore, the integral points of $GT_{\lambda}$ parameterize a Gelfand-Tsetlin basis of the $GL_n$-module with highest weight $\lambda$, so the number of integral points equals the dimension of this module. GT polytopes can also be viewed as the marked order polytope of a poset as discussed in [1].

Theorem 1 (Diameter of 1-skeleton)

It suffices to consider $\lambda = (1^n, \ldots, m^m)$ for $a_i \in \mathbb{Z}_{\geq 0}$. For any $GT_{\lambda}$, the diameter of the 1-skeleton is $\text{diam}(GT_{\lambda}) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

Theorem 2 (Automorphism Group)

It suffices to consider $\lambda = (1^n, \ldots, m^m)$ for $a_i \in \mathbb{Z}_{\geq 0}$. $m = 2$. Suppose $\lambda = (1^n, 2^m)$ and $a_1, a_2 \geq 2$. Then

$\text{Aut}(GT_{\lambda}) \cong D_4 \times Z_2$.

Otherwise,

$\text{Aut}(GT_{\lambda}) \cong D_4 \times Z_2 \times Z_2^{\delta_{1,a_1}}$, where $D_4$ is the dihedral group of order 8 and $Z_2$ is the cyclic group of order 2.

Proof Idea for Theorem 2

First we exhibit a set of automorphisms and show that they generate the groups in Theorem 2.

Noting that facets in a GT-polytope are in bijection with single edges in $GT_{\lambda}$, we represent facets by their associated edge. Two facets are called dependent if their intersection is $d - 3$ dimensional. We partition the edges of $GT_{\lambda}$ into maximal chains of dependent facets.

For any $\phi \in \text{Aut}(GT_{\lambda})$ and chains $C_1, C_2$.

• If $\phi(C_1) = C_2$, then $C_1$ is mapped to $C_2$ or its flip.

• $\phi$ preserves the lengths of chains.

• $\phi$ preserves adjacency of chains.

Starting with the facets in chains of length 1 and length 2, we bound the size of the orbits of these facets and iteratively apply the Orbit-Stabilizer Theorem. We show that the order of the automorphism group equals the order of the groups in Theorem 2.

References


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