The weak Bruhat order on the symmetric group is Sperner

Yibo Gao
Joint work with: Christian Gaetz

Massachusetts Institute of Technology

FPSAC 2019
Overview

1. The Sperner property of weak Bruhat order
   - The Sperner property of Posets
   - An $\mathfrak{sl}_2$-action on the weak Bruhat order of $S_n$
   - Open problems

2. Further work related to the code weights
   - A determinant formula by Hamaker, Pechenik, Speyer and Weigandt
   - Padded Schubert polynomials
   - Weighted enumeration of chains in the (strong) Bruhat order
The Sperner property

Let $P$ be a ranked poset with rank decomposition $P_0 \sqcup P_1 \sqcup \cdots \sqcup P_r$.

**Definition**

$P$ is called $k$-Sperner if no union of its $k$ antichains is larger than the union of its largest $k$ ranks.

$P$ is called Sperner if it is 1-Sperner.

$P$ is called strongly Sperner if it is $k$-Sperner for any $k \in \mathbb{Z}_{\geq 1}$.

*Figure:* A Sperner poset (left) and a non-Sperner poset (right)
The Sperner property

Further assume that \( P = P_0 \sqcup \cdots \sqcup P_r \) is

- rank symmetric: \( |P_i| = |P_{r-i}| \) for all \( i \),
- rank unimodal: there exists \( m \) such that
  \[
  |P_0| \leq |P_1| \leq \cdots \leq |P_m| \geq \cdots \geq |P_{r-1}| \geq |P_r|.
  \]

**Definition**

An order lowering operator is a linear map \( D : \mathbb{CP} \to \mathbb{CP} \) such that

\[
D \cdot x = \sum_{y \leq x} \text{wt}(y, x) \cdot y, \quad x \in P_i.
\]

**Figure:** An example of an order lowering operator.
Recall $P = P_0 ⊔ \cdots ⊔ P_r$ is rank symmetric and rank unimodal.

**Lemma (Stanley 1980)**

If there exists an order lowering operator $D$ such that

$$D^{r-2i} : \mathbb{C}P_{r-i} \to \mathbb{C}P_i$$

is an isomorphism for any $0 \leq i \leq \lfloor r/2 \rfloor$, then $P$ is strongly Sperner.

Together with the hard Lefschetz theorem in algebraic geometry, Stanley proved the following:

**Theorem (Stanley 1980)**

Let $(W, S)$ be a Coxeter system for which $W$ is a Weyl group. Then the (strong) Bruhat order on $W$ or any parabolic quotient $W^J$ is rank symmetric, rank unimodal and strongly Sperner.
The Sperner property (via $\mathfrak{sl}_2$ representations)

Definition
An $\mathfrak{sl}_2$ representation on $P$ consists of the following data:

- an order lowering operator $D : \mathbb{CP}_i \rightarrow \mathbb{CP}_{i-1}$, $\forall i$,
- a raising operator $U : \mathbb{CP}_i \rightarrow \mathbb{CP}_{i+1}$, $\forall i$,
  (U doesn’t need to respect the order)
- a modified rank function $H : \mathbb{CP}_i \rightarrow \mathbb{CP}_i$, $x \mapsto (2i - r)x$,

such that $UD - DU = H$.

In fact, $U, D, H$ make $\mathbb{CP}$ an $\mathfrak{sl}_2$ representation.

Theorem (Proctor 1982)
A ranked poset $P$ admits an $\mathfrak{sl}_2$ representation if and only if $P$ is rank symmetric, rank unimodal and strongly Sperner.
The weak and strong Bruhat orders (on $S_n$)

For $w \in S_n$, let $\ell(w)$ denote the usual Coxeter length. The (right) weak (Bruhat) order $W_n$ is generated by

$$w \leq_W w s_i \quad \text{if} \quad \ell(ws_i) = \ell(w) + 1, \text{ where } s_i = (i, i + 1).$$

The (strong) Bruhat order $S_n$ is generated by

$$w \leq_S w t_{ij} \quad \text{if} \quad \ell(w t_{ij}) = \ell(w) + 1, \text{ where } t_{ij} = (i, j).$$

![Diagram](image)

**Figure:** The weak and strong order on $S_3$. 

The weak order is Sperner
The weak and strong Bruhat orders (on $S_n$)

Stanley (1980) showed that the strong Bruhat order (on any Weyl group) is strongly Sperner, and has a symmetric chain decomposition for types $A_n, B_n, D_n$.

Björner (1984) conjectured that the weak Bruhat order is strongly Sperner.

Stanley (2017) suggested an order lowering operator

$$D \cdot w = \sum_{\ell(ws_i) = \ell(w) - 1} i \cdot (ws_i).$$

Conjecture (Stanley 2017)

For $D$ defined as above, $D\binom{n}{2}^{-2i} : \mathbb{C}(W_n)_{\binom{n}{2} - i} \to \mathbb{C}(W_n)_i$ has nonzero determinant for $0 \leq i \leq \binom{n}{2}/2$. Thus, the weak Bruhat order $W_n$ is strongly Sperner.
An $\mathfrak{sl}_2$ action on the weak Bruhat order $W_n$

**Proposition (Gaetz and G. 2018)**

The following data give an $\mathfrak{sl}_2$ action on $W_n$:

- **the order lowering operator suggested by Stanley**
  \[
  D \cdot w = \sum_{\ell(ws_i) = \ell(w) - 1} i \cdot (ws_i),
  \]

- **a raising operator defined by**
  \[
  U \cdot w = \sum_{w \leq_S u} \|\text{code}(w) - \text{code}(u)\|_{L^1} \cdot u,
  \]

- \(H \cdot w = (2\ell(w) - \binom{n}{2}) \cdot w.\)

Recall \(\text{code}(w)_i = \{j > i : w(j) < w(i)\}\).
An $\mathfrak{sl}_2$ action on the weak Bruhat order $W_n$

**Figure:** The order lowering operator $D$ and the raising operator $U$

The (unique) raising operator $U$ that corresponds to $D$ doesn’t need to be supported on the strong order. It’s just nice combinatorics.

**Corollary (Gaetz and G. 2018)**

The weak order $W_n$ on the symmetric group is strongly Sperner.
Open Problems

Conjecture

The weak Bruhat order is strongly Sperner for any Coxeter group.

Conjecture

The weak Bruhat order of type $A$ has a symmetric chain decomposition.

Example (Leclerc 1994)

The weak order of $H_3$ doesn’t have a symmetric chain decomposition, but is strongly Sperner.
Hamaker, Pechenik, Speyer and Weigandt resolved the full determinant conjecture by Stanley.

**Theorem (Hamaker et al. 2018, conjectured by Stanley 2017)**

\[
\det D^{(n)_2-2k} = \left( \binom{n}{2} - k \right)! \#(W_n)_k \prod_{i=0}^{k-1} \binom{n-k-i}{k-i} \#(W_n)_i
\]
Definition (Schubert Polynomials)

The Schubert Polynomials $S_w$, for $w \in S_n$, can be defined as follows:

- $S_{w_0} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}$,
- $S_w = \partial_i S_{ws_i}$ if $\ell(w) = \ell(ws_i) - 1$,

where $\partial_i f = (f - s_i f)/(x_i - x_{i+1})$ is the $i$th divided difference operator.

Proposition (Hamaker et al. 2018)

Let $\nabla = \sum_i \partial/\partial x_i$. Then

$$\nabla S_{w^{-1}} = \sum_{i: \ell(w) = \ell(ws_i) + 1} i \cdot S_{s_i w^{-1}}.$$

Corollary (Macdonald’s Identity)

$$\sum_{\text{reduced } s_{a_1} \cdots s_{a_N} = w_0} a_1 \cdots a_N = \binom{n}{2}!.$$

The weak order is Sperner
Recall that \( \{ \mathcal{S}_w \}_{w \in S_n} \) form a basis of \( \text{span}_\mathbb{C}\{x^\alpha : \alpha \leq \rho\} \) where \( \rho = (n-1, \ldots, 1) \) is the staircase partition.

**Definition (Gaetz and G. 2018)**

The *padded Schubert polynomial* \( \tilde{\mathcal{S}}_w \) is the image of \( \mathcal{S}_w \) under

\[
x^\alpha \mapsto x^\alpha y^{\rho-\alpha}.
\]

Define the following linear operators

\[
\nabla = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} y_i, \quad \Delta = \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} x_i.
\]

**Proposition (Hamaker et al. 2018; Gaetz and G. 2018)**

1. \( \nabla \tilde{\mathcal{S}}_{w^{-1}} = \sum_{i : \ell(w) = \ell(ws_i) + 1} i \cdot \tilde{\mathcal{S}}_{s_i w^{-1}}. \)
2. \( \Delta \tilde{\mathcal{S}}_{w^{-1}} = \sum_{u : u \geq_s w} \| \text{code}(u) - \text{code}(w) \|_{L^1} \cdot \tilde{\mathcal{S}}_{u^{-1}}. \)
We see that

\[
\left( \sum \frac{\partial}{\partial y_i} x_i \right) (x_1 y_1 y_2 + y_1^2 x_2) = 3x_1 y_1 x_2 + x_1^2 y_2.
\]
Weights on the strong Bruhat order

Let $a_{w \preceq u} = \{k < i : w(i) < w(k) < w(j)\}$ and similarly define $b_{w \preceq u}$, $c_{w \preceq u}$ and $d_{w \preceq u}$.

For example, when $w = 4127653$, $u = 4157623$,

$$a_{w \preceq u} = 1, \quad b_{w \preceq u} = 2, \quad c_{w \preceq u} = 1, \quad d_{w \preceq u} = 0.$$
Weighted enumeration of maximal chains

If \( \text{wt} : E \to R \) is a weight function on covering relations, where \( R \) is a commutative ring, we can define, for \( x \leq y \),

\[
m_{\text{wt}}(x, y) = \sum_{C : x \to y} \prod_{e \in C} \text{wt}(e).
\]

**Theorem (Gaetz and G. 2019)**

Let \( z_A, z_B, z_C, z_D \) be indeterminates and define a weight function on the covering relations on the strong Bruhat order of \( S_n \) as follows:

\[
\text{wt}(w \lessdot u) = 1 + z_A a_{w \lessdot u} + z_B b_{w \lessdot u} + z_C c_{w \lessdot u} + z_D d_{w \lessdot u}.
\]

Then if \( \{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2 - z\} \) as multisets,

\[
m_{\text{wt}}(\text{id}, w_0) = \binom{n}{2}!.
\]
Weighted enumeration of maximal chains

Let \( w \triangleleft u \) = \( 1 + z_A a_{w \triangleleft u} + z_B b_{w \triangleleft u} + z_C c_{w \triangleleft u} + z_D d_{w \triangleleft u} \).

Then \( m_{\text{wt}}(123, 321) = 4 + z_A + z_B + z_C + z_D \), which is 6 = 3! if \( \{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2 - z\} \).

Figure: Weights on covering relations of \( S_3 \)
Weighted enumeration of maximal chains

**Theorem (Gaetz and G. 2019)**

Let \( z_A, z_B, z_C, z_D \) be indeterminates and define a weight function on the covering relations on the strong Bruhat order of \( S_n \) as follows:

\[
\text{wt}(w \lessdot u) = 1 + z_A a_{w \lessdot u} + z_B b_{w \lessdot u} + z_C c_{w \lessdot u} + z_D d_{w \lessdot u}.
\]

Then if \( \{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2 - z\} \) as multisets,

\[
m_{\text{wt}}(\text{id}, w_0) = \binom{n}{2}!.
\]

Special cases:

1. \((z_A, z_B, z_C, z_D) = (0, 1, 0, 1), \) \( \text{wt}(w \lessdot \text{wt}_{ij}) = j - i, \)
2. \((z_A, z_B, z_C, z_D) = (0, 0, 2, 0), \) \( \text{wt}(w \lessdot u) = \|\text{code}(w) - \text{code}(u)\|_{L^1}. \)

The “\( j - i \)” weight is commonly known as the Chevalley weight, which is investigated by Stembridge (2002) and further by Postnikov and Stanley (2009). It is still open to find a combinatorial proof.


Thanks: Alex Postnikov and Richard Stanley.

Thank you for listening!