Study: Lagrangian Surfaces in the 4-Torus (just a M. Abouzaid).

Question: Describe Lagrangian surfaces in the standard \((T^4, w_{std}) = (R^4/Z^4, w_{std})\) of Maslov class zero.

Example: There are no Lag in 2-spheres.

Proposition: Let \(L \subset T^4\) be a Lagrangian torus of \(\mu_2 = 0\). Then \(L\) is Floer cohomologically indistinguishable from a linear Lagrangian torus. In particular, \(\text{Coh} \cdot H^*_c(T^4; \mathbb{Z})\) is primitive.

Ranks: (1) In fact, argument applies to \(T^n \subset T^{2n}\).

(2) \(S^1 \subset T^2\) embedded, \([S^1]\) is primitive, e.g. through of geodesic representative.

Note: View \(T^4\) as a trivial \(T^2\)-bundle over \(T^2\), as a Lagrangian fibration. We can form \(T^2\) (following L. Polterovich) the Lagrangian surgery, a Lagrangian \(\Sigma_2 \hookrightarrow T^4\), cohom.

Claim (Abouzaid/IS): Any Lagrangian \(\Sigma_2 \hookrightarrow T^4\) of Maslov class zero is Floer cohomologically indistinguishable from such examples.

Cor: Given such an \(L \subset \Sigma_2\), then there is some \(T^2\)-fibre \(F \subset T^4\) s.t. \([L']\) is homotopic to \(L \otimes L'\) if \(L, L'\) then \(L' \cap F \geq 3\) pts.

Naive style of argument when \(L = T^2\): -

2 cases - \(L \hookrightarrow T^4\) is or isn't \(w_1\)-injective.

If it is, \([L] \neq 0\), with an underlying primitive homology class \([L]\) spanning some linear Lagrangian torus \(R\).

Unwind directions orthogonal to \(R\): Lift \(L\) to \(T^*R\), as some (infinite union) of disjoint Lagrangian tori.
Let $\mathcal{T}$ be any such. $T = \{e\} \neq \emptyset$. 

Then $\mathcal{T} \in H^1(\mathcal{T}, \mathbb{R})$ comes from a unique elt. $\psi e H^1(\mathcal{T}, \mathbb{R})$.

If we view $\mathcal{T}$ as flux of translation $\mathcal{T}$, defines another torus $\mathcal{T}' = T \times \mathbb{R}$, so from view of $T \times \mathbb{R}$, $\mathcal{T}$ looks exact $\Rightarrow \mathcal{T}$ is homological $\mathbb{R}$.

But if $L \to L'$ wasn't $\pi$, injective, then lift $L$ to some $L \in \Lambda$, every space of $T^4$ which is subtorus. This would violate standard results of Fukaya et al. (Sym).

Strategy to prove results for higher genus surfaces:

- Use HMS for the 4-torus.

Suppose that $D\mathcal{T}^4(T^4) = \mathcal{DCH}_{\mathcal{T}}(T^4)$

Fukaya et al. Derived cat. of coherent sheaves of $(T^4, \omega_{T^4})$ on some abelian surface.

Then $L \in T^4$, $L$ defines an orb. $P^*$ complex of sheaves of lifts of $L$ stabilizes sheaves inside $L$.

Recall: Since $P^*$ is smooth to $\mathcal{E}_2$, $H^k(\mathcal{E}_2) \cong H^k(L, L) \cong \mathcal{E}_2^{k+2}(P^*, P^*)$

on the other hand, $\mathcal{E}$ spectral sequence

\[ E^2_{p,q} \Rightarrow \mathcal{E}_{p+q} = \bigoplus \operatorname{Ext}^p(H^d, H^q) \Rightarrow \operatorname{Ext}^{p+q}(P^*, P^*) \]

\[ k- j = 2 \]

column sheaves of complex $P^*$

Spectral seq. $\Rightarrow \operatorname{Ext}^1(P^*, P^*) \geq \bigoplus \operatorname{Ext}^{p+1}(H^d, H^q)$

For any coherent sheaf on an orb. surface, Ext has rk at least 2 $\Rightarrow P^*$ is sheaf lor length 2 complex.
Case where \( \exists \) 2 cohomologies, \( H^0(P) \) \& \( H^1(P) \)

Claim: each of these 2 sheaves is simple, i.e. its

\[ \text{Ext}^0(H^1, H^1) \] is rank one (i.e. \( P^* \) looks like an
extension of 2 objects w/ morphisms \( H^0(T^2) \)).

If not simple, then not stable.

Our sheaf \( E \) -

\[
0 \rightarrow \text{Tors} \rightarrow E \rightarrow E_0 \rightarrow 0
\]

Hence, Numerical filtration of \( E_0 \)

\[
G_0 \leq G_1 \leq \ldots \leq G_n = E_0, \ G_i/G_{i-1} \text{ semi-stable.}
\]

Specify sequence

\[
\text{Ext}^i(E, E) \cong \text{Ext}^i(\text{Tors}, \text{Tors}) \oplus \text{Ext}^i(\mathbb{Q}_i, \mathbb{Q}_i)
\]

If any \( E \) is semi-stable, \( E \) underlying stable object,

\[
\sum (-1)^i \text{Ext}^i(E, E) = \sum (-1)^i \text{Ext}^i(\text{Stable, Stable})
\]

\( \Rightarrow \neq \ \checkmark \) CHECK!

Central issue: Do we know HMS for \( T^2 \times T^2 \)?

Partial results:

- On abelian varieties, Fukaya relates Lagrangians in

  "good position" to sheaves

- Work of Kontsevich \& Soibelman building

  \( D^b \text{Coh}(A) \rightarrow D^b \text{Fuk}(A^*) \), not obvious that the image generates

General strategy for showing some objects generate a Fukaya algebra

"Resolution of the diagonal" := if \( \{ v_i \} \) a collection of vectors in

a vector space, then if \( \{ v_i \} \) \& \( v \) write id = \( E_i : v_i \rightarrow v \),

Some \( E_i \), then \( v \) span.
Technical Statement:

Let $A \leq \mathcal{F}(Z)$ be a full subcategory. For objects $k_+ \in A$, $H$ projection functor $L \mapsto (k_+, L) \otimes k_+ \in \text{Tw}(A)$. (Technical hypothesis: $\text{el} A$ bounded no hol. discs)

If (1) the natural map $QH^*(Z) \to HH^*(A)$ is an iso, AND

(2) if $A \in \text{Subcat of Fun}(A_A, A)$ generated by projection functors

$\Rightarrow A \to \mathcal{F}(Z)$ split-generates the Fukaya category.

Think of: $Z = T^2 \times T^2$, say $A = \{L_i \otimes L_j | L_i, L_j \text{ generate } \mathcal{F}(T^2) \text{ coming from } \mathcal{F}(T)\}$

Naive idea was $\mathcal{F}(Z, Z) \to \text{Fun}(\mathcal{F}(Z), \mathcal{F}(Z))$

using naive idea that $f \leq Z \times Z$ Lag correspondence, then given $L \leq Z$, look at

$\exists w \in Z | \exists \alpha, L \in Z \text{ s.t. } (w, \alpha) \leq L$

This is generally only immersed!

Man-Wilsh-Vans-Woodward: Introduce $\mathcal{F}^*(Z)$ a category of generalized Lagrangians.

$pt. \xrightarrow{L_0} M_1 \xrightarrow{L_{12}} M_2 \xrightarrow{L_{21}} Z$.

There is a natural functor $\mathcal{F}^*(Z, Z) \to \text{Fun}(\mathcal{F}^*(Z), \mathcal{F}^*(Z))$

Introduce $A^* = \mathcal{F}^*(A)$, pt. $\xrightarrow{f} \mathcal{F}^*(Z, Z) \to Z \to Z \to \ldots$

$\mathcal{F}(Z, Z) \to \mathcal{F}(Z, Z) \to \mathcal{F}^*(Z, Z)$
Key Lemma: For $K^*_2 \in A$, $K = K^- \times K^+$,
$\hat{\Phi}_{mww}(K) \cong I(K^*_2)$ are $iso^2$ $A_{st}$-functors.
$\gamma$ $\Phi$ restricts to $\hat{\Phi} : A(Z \times Z) \to \text{End} (Tw A)$

Sum-up:

$\mathcal{F}(Z_2 \times Z_2) \to \text{Fun} (\mathcal{F}_4 \times Z_2, \mathcal{F}_4 \times Z_2)$

$\mathcal{F}(Z_2 \times Z_2) \to \text{Fun} (A \otimes Z_2, A \otimes Z_2)$

Use the technical statement working on $Z$.  

$\mathcal{F}(Z_2 \times Z_2)$

12 knowledge of $Z_2$

$\text{Fun} (\mathcal{F}_4 \times Z_2, \mathcal{F}_4 \times Z_2)$

12 $\Phi$ MS

$\mathcal{F}(Z_2 \times Z_2)$

$\mathcal{F}(\text{Coh} Z_2, \text{Coh} Z_2)$

$D^{-1} \text{Coh} (Z_2 \times Z_2)$