Seidel—Lag'n Tori in the quartic surface

\[ \text{McCp}^3 \text{ quartic surface.} \]

comes with a symplectic form \( \omega = \psi(x, \text{vol. form. LCM oriented Lag'n surface.} \]

\[ [\mathcal{L}] \in H_2(M; \mathbb{Z}) \quad \mathcal{L}_L \in H^1(L; \mathbb{Z}) \]

\[ [\mathcal{L}]^2 = -\chi(L) \]

Ex: \( T^2 \subset \mathbb{C}^2 \) cm Clifford torus, \([\mathcal{L}] = 0 \) but \( \mathcal{L}_L \neq 0 \).

Ex: \( \text{LCM special Lag'n, i.e., } \mathcal{L}_L = 0 \) but \([\mathcal{L}] \neq 0 \)

\( \text{(with integrating hol. vol. form gives } \text{area} \neq 0 \). \)

Thus: If \( \mathcal{L}_L = 0 \), then \([\mathcal{L}] \neq 0 \).

Actually a theorem about tori. (WHY?)

Related Conj.: Let \( \phi \in \text{Aut}(M, \omega) \), \( \phi_* = 1 : H_2(M; \mathbb{Z}) \)

and \( \text{Fix}(\phi) \) nondeg., then \( |\text{Fix}(\phi)| \geq \sum \text{dim } H_i(M; \mathbb{Q}) \)

Proof by meshed thinking: Take \( T^2 \subset L \subset \mathbb{C}^2 \), \( \mathcal{L}_L = 0 \),

apply the mean curvature flow,

\[ L \xrightarrow{\text{singularity}} \text{limits are special Lag'n—meaning).} \]

Deform \([\mathcal{L}] \) slightly so that \([\mathcal{L}] = H_2(M; \mathbb{Z}) \neq 0 \).

If \([\mathcal{L}] = 0 \), torus survives, but \( \not\in \text{SLAG} \).

This doesn't actually work!

(Comes from analogy between SLAG & stable bundles? Only holds in large comp. struct. limit...)
A deformation of \( \mathcal{A} \) over field \( \mathbb{A} \) is a curved \( \mathcal{A} \) category \( \mathcal{A} \)
\[
\text{Ob} \mathcal{A} = \text{Ob} \mathcal{A}^o
\]
\[
\text{hom}_{\mathcal{A}}(Y_0, Y_1) = \text{hom}_{\mathcal{A}^o}(Y_0, Y_1) \otimes R.
\]
\[
\mathcal{A}^{\circ}_A = \mathcal{A}^o_A + O(h)
\]
This deformation theory is governed by a dg Lie algebra \( \mathcal{C}^*(\mathcal{A}, \mathcal{A})[1] \). In particular, the \( N=1 \) deformations are classified by \( HH^2(\mathcal{A}, \mathcal{A}) \).

Note that we have a curvature term
\[
\mathcal{A}_A^{\circ} \in \text{hom}_{\mathcal{A}^o}(Y, Y) \text{ of order } O(h).
\]

Define a new \( \mathcal{A} \) category \( \mathcal{A} \) with no curvature. Objects of \( \mathcal{A} \) are pairs
\[
(Y \in \text{Ob} \mathcal{A}, \chi \in \text{hom}_{\mathcal{A}}^1(Y, Y), \quad \mathcal{A}_A^{\circ} + \mathcal{A}^1_A(\chi) + \mathcal{A}^2_A(\chi^2), \quad \chi = O(h)).
\]

Remark: We'll also use deformations over \( R = \bigwedge [C[t]] \), which are best treated as inverse limits.

Homological Mirror Symmetry:

The mirror of \( M \) is a K3 surface \( X \) over \( \mathbb{C}((t)) \).

This means that we have a quasi-isomorphic embedding
\[
\mathcal{A} = \mathcal{F}(M) \to \mathcal{B} \quad (\mathcal{B} \text{ dg cat underlying } \text{Db} \text{Cat}(X))
\]
which induces an equivalence \( T^w \mathcal{X} = \mathcal{B} \) ("Bar-Summerville orbits")

As a consequence \( HH^*(A,A) \cong HH^*(B,B) \) and in fact we have an identification of deformation theories \( (HH \text{ mv. under } T^w) \). So def. of \( A \) \( \leftrightarrow \) def. of \( B \), and for the associated categories \( \widetilde{A} \leftrightarrow \widetilde{B} \).

Symplectic case
\[
H^*(M) \rightarrow HH^*(A,A) \text{ is an isomorphism}
\]
(in particular, \( H^2(M) \rightarrow HH^2(A,A) \text{ iso.} \))

\( \) deformation class?

Take \( LCM \) with \( \alpha_L = 0 \) and \( [L] = 0 \). This will survive into any deformation, i.e., we always get a \( \mathbb{C} \) Ob \( \mathcal{A} \) such that \( HH(\text{change}(\mathcal{A},\mathcal{L})) = HH(\text{change}(\mathcal{A},\mathcal{L})) \otimes \mathbb{C} \).

Deformation for chain complexes \( = H^2(T^2; \mathbb{C}) \).

\( \) (Lowen-van der Bergh)
\[
\text{Coh}(X) \subset \text{QCo}(X) \supset \text{Inj}(X)
\]

\( \) abelian categories injective.

\( B \) = dg category ofbdd. below complexes of injective sheaves with bounded coherent cohomology.

All deformations of \( B \) come with a deformation of the entire package:

\[
\text{Coh}(X) \subset \text{QCo}(X) \supset \text{Inj}(X)
\]

\( \) abelian categories objects flat over \( R \) and become injective after reduction to \( h = 0 \).
And indeed, $B$ is bounded below complexes of
$\text{Inj}(X)$-objects, with conditions as before (e.g. coherent sheaves bounded, live in other guys).

Take $F \in \text{Ob} \text{CoH}^\cdot(X)$. Set
\[
F = \tilde{F}/h F, \text{ or } \tilde{f}(i) = h^i \tilde{F} / h^i h \tilde{F}
\]
Then $F \in \text{Ob} \text{CoH}^\cdot(X) \cap \text{Ob} \text{CoH}^\cdot(X)$.

These come with surjective maps $\tilde{f} \xrightarrow{h} f(i) \xrightarrow{h} f(2i)$.

The first order class $b \in H^0(B, B)$ determines for each $F$ an obstruction class $\phi \in \text{Ext}^2_{\text{CoH}^\cdot(X)}(F, F)$.

If $\phi \neq 0$, $F \xrightarrow{h} f(i)$ can't be an isomorphism.

Similarly, if $\phi(i) \neq 0$ then $f(2i) \xrightarrow{f(i)} f(i+1)$ is not an iso.

Note: In $\text{CoH}^\cdot(X)$, every chain of surjections must readily become isomorphisms. (Not true for injectives).

Specifically for $k3$ surfaces, the following holds:

If $b$ is chosen generically, then $g = 0 \forall \neq 0 F$.

Thus for all $F \in \text{Ob} B$, we have $H^m F = 0$ for $m > 0$.

By a standard spectral sequence,
\[
\forall 2 \in \text{Ob} B, H^2(\text{hom}_X(\tilde{E}, \tilde{E})) \text{ is a torus},
\]

This actually proves the theorem by contradicting $R$-module.

$A \rightarrow B$.

$Coh(X) = Coh(X)$? on complex side,

$H^2(B, B) = H^0(X, \wedge^2 TX) \otimes H^1(X, TX) \otimes H^2(X, O_X)$
Given any $b \in H^2(B, B)$ and any $F \in OB_c(X)$ we have $c_F \in Ext^2(F, F) = H^2(X; F \otimes \mathcal{F}^*)$.

Formula

\[ \langle [\mathcal{F}], b \rangle = \langle ch(F), b \rangle \]

\[ H^2(X; \mathcal{O}_X^*) \]

\[ H^*(X; \Omega_X^*) \]

\[ H^*(X, \Lambda^TX) \]

(Illusie)

Case 1: $F$ vector bundle. Take $0 \neq b \in H^2(X, \mathcal{O}_X) \cong \Lambda$.

\[ \langle ch(F), b \rangle = \text{rank}(F) b \neq 0. \]

Deformations to twisted sheaves (modifies cycle condition by $e^b$ — kills structure sheaf).

Case 2: $F$ is supported at a point, take

\[ 0 \neq b \in H^0(X, \Lambda^2 TX) \cong \Delta \]

(dual of vol. form, i.e. Pontrjagin)

\[ \langle ch(F), b \rangle = \langle c_1(F), b \rangle \neq 0. \]

Deformations to non-commutative variety.

Case 3: $F = \mathcal{O}_c$ structure sheaf of a curve, take

\[ b \in H^1(X, TX). \]

\[ \langle ch(F), b \rangle = \langle c_1(F), b \rangle \neq 0 \text{ for generic disk of } b. \]

These are the deformations.