Ng-Legendrian (Rot'd) Symplectic Field Theory II
(for knots)

Today: Setup, problem
Tomorrow: Solution, algebraic details
Combina...rial?
Wednesday: knot contact homology + string topology
relevant preprint: 0806. 4598
motivational work: joint w/ Cieliebak, Elenkin, Latschev.

Goal: define rational relative SFT for Legendrian knots.
Motivation: develop relative side of SFT
- produce holomorphic invariant of Legendrian knots
  (generalizing contact homology
  - transverse knots (cf. knot Floer homology)
  - topological knots
  - later: generalize to higher dim'l contact manifolds
    (general construction - knot invariants)
  - Legendrian surgery?...

General setup \((V^{2n+1}, \xi)\) contact manifold, \(\xi = \ker \alpha\)
\(U = \alpha \in \Omega^1(V), \alpha \wedge (d\alpha)^n = \text{vol. form}\)
\(L\) : Legendrian if \(L\) is everywhere tangent

Ex: \(V = \mathbb{R}^3, x_{\text{std}} = \ker (dt - ydx)\)
\(V \times \mathbb{R}_+\) symplectization, \(\omega = d(e^t \alpha)\)
Look at holomorphic curves in $V \times \mathbb{R}$ with boundary on $\Lambda \times \mathbb{R}$. Maximum principle: no nonconstant compact curves.

But what a.e. structure to use on $V \times \mathbb{R}$? J a.e. structure preserves contact planes on each slice, $J(\partial_3) = \partial_x$, $J(\partial_2) = \partial_x$. Reeb v.f., vertically invariant.

Reeb v.f. $\mathbb{R}$ on $V$ is defined by $\mathbb{R}$-invariant:

$\mathbb{R} - \delta x = 0$

$\mathbb{R} (\mathbb{R}) = 1$

e.g. $(\mathbb{R}^3, \mathbb{R} = dz - y dx)$, $\mathbb{R} = \frac{a}{\partial z}$.

Reeb chord = Reeb flow beginning and ending in $\Lambda$.
Reeb strip ($\text{Reeb chord} \times \mathbb{R}$) is totally holomorphic.

In general, consider hol. curves w/ boundary punctures approaching Reeb strips at $+\infty$.

First, count holomorphic steps:

- $\mathbb{R}$ dimension $d$ of $\mathbb{R}$-action
- $\mathbb{R}$-invariant $\Lambda \times \mathbb{R}$
- $\mathbb{R}$-invariant Reeb
- $\mathbb{R}$-invariant Reeb strip
- $\mathbb{R}$-invariant Reeb
- $\mathbb{R}$-invariant Reeb strip (like Morse theory for affine structure on Path Spaces)
Problem 1: This Mark theory predicts: $d^2 = 0$!

So naturally forced to consider hol. curves w/ arbitrary #s of punctures.

Level 1 SFT: hol. disks w/ arbitrary #s of bd. punctures but exactly one $\to \pm \infty$

Relative Contact Homology

Level 2 SFT: hol. disks w/ any # of punctures $\to \pm \infty$.

E.g.

Level 3 SFT: arbitrary genus hol. curves w/ punctures on boundary.

Level 4 SFT: add marked pts., -- more decodings

Algebraic formalism: quasigenerically $\Lambda$ has fin. many Reeb chords $\langle V, x, \Lambda \rangle \to R_{1..m} \to \mathbb{R}$

For $\{1, 2, \ldots, q_m \}$

E.g. 1-jet spaces $T^\ast M \times R$, we assume no closed Reeb orbit associates formal invariants.
Define \( M(p_i) \): 
\[
M(p_i, p_{i_1}, 2i_3, 9i_4) = \text{moduli space of punctured hol. disks in } V \times IR \text{ w/ bd. in } \Lambda \times IR \text{ and successive punctures} \rightarrow \text{Reeb strips at } p_i, p_{i_1}, 
\]
e.g. 
\[
\begin{array}{c}
p_{i_2} \\
\end{array} \rightarrow 
\begin{array}{c}
p_{i_2} \\
\end{array} 
\]
\[
\begin{array}{c}
p_{i_4} \\
\end{array} 
\begin{array}{c}
p_{i_3} \\
\end{array} 
\]
Note dearly invariant under cyclic permutation. (missing over \( \mathbb{Z}/2 \) here...)

\( \text{Note } M(z, z', z', z') = \aleph \text{ arbitrary punctures} \)

by maximum principle.

First consider \( M \) s w/ one \( \mathcal{P} \) (will give us \( CH \)).

\( \text{Compatitive, unital i.e. empty word } \)
\( A_{CH} = \text{tensor algebra gen (over } \mathbb{Z}/2 (\mathbb{Z}) \text{) by } z_i, \bar{z}_i \) 

Define \( dh \text{ on } A_{CH} \) by:

\[
dh(z_i) = \sum \left( M(\cdot)/R \right) z_{j_1}, \ldots, z_{j_m} 
\]

\[
\text{dim } M(p_i, 2i_{j_1}, \ldots, 2i_{j_m}) = 0 \quad (\forall) x \geq 0. 
\]

\( \text{Extend by linearity to all of } A_{CH}. \)

Can grade \( A_{CH}: \text{to each } R_i \rightarrow \text{"associate" Conway-Zink index } CZ(R_i) \) (actually need some data, a "jumping path"?)

Index formula: Generally

\[
\text{dim } M(p_i, 2i_{j_1}, \ldots, 2i_{j_m}) = (CZ(R_j) - 1) \cdot \left( \sum_{k=1}^{m} (CZ(R_{j_k})) \right) 
\]

If we set \( |z_{j_k}| = CZ(R_{j_k}) - 1 \), then \( \delta \) lowers \( \text{dim } M(\cdot) \) degree by 1.
Thus:

Eliashbaer, Chekanov for $\mathbb{R}^2$, Ekholm-Etnyre-Sullivan for e.g. 1-Jets spaces $\mathbb{P} \times \mathbb{R}$, etc.

$d^2 = 0$ and the homotopy equivalence type of $(\mathbb{A}, \mathbb{H})$ is an invariant of the Legendrian $\mathbb{A}$.

$H_s(\mathbb{A}, \omega)$ = contact homology

$x \geq proj:

\begin{tikzpicture}
\begin{scope}[scale=0.5]
\draw [thick] (0,0) -- (2,2) -- (4,0) -- cycle;
\draw [thick] (2,2) -- (4,4) -- (6,2) -- cycle;
\draw [thick] (0,0) -- (0,4);
\draw [thick] (2,2) -- (2,4);
\draw [thick] (4,0) -- (4,4);
\draw [thick] (6,2) -- (6,4);
\end{scope}
\end{tikzpicture}

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\draw [thick] (0,0) -- (2,2) -- (4,0) -- cycle;
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\draw [thick] (0,0) -- (0,4);
\draw [thick] (2,2) -- (2,4);
\draw [thick] (4,0) -- (4,4);
\draw [thick] (6,2) -- (6,4);
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\end{scope}
\end{tikzpicture}

Proof of $d^2 = 0$:

$d^2(q_i) = \cdots$:

$A$ = tensor alg. germ'd by $\mathbb{P}$, $\mathbb{E}$, $\mathbb{Q}$, $\mathbb{R}$

$d(q_i) = \sum q_i \otimes q_i$

$d(p_i) = \sum q_i 

\text{Level 1}:

d(\xi_i) = \sum q_i$
So: let \( A_{cyc} = A/[A,A] \) be the Z/2
module gen'd by cyclic words in \( p^i, q^j \).

Want to construct a differential on \( A_{cyc} \).

\[ d^2 = 0 \]

\[ \rightarrow \text{contact contact homology} \]