Seidel Day IV Part II

Wrapped Floer Cohomology

\[ p : E \to C \text{ Lefschetz fibration} \]

\[ \text{a properly embedded } \xi \text{ vanishing path. This determines a properly} \]

\[ \text{embedded noncompact Lefschetz thimble } \mathbb{R}^{n+1} \cong L \subset E. \]

We want to consider the wrapped Floer cohomology \( \text{HF}^*(L,L) \). Take the diffeomorphism of \( C \) which is the right-handed Dehn twist along a large circle, and lift it to a symplectomorphism of \( E \), called \( \eta : E \to E \).

\[ \text{over here, have the global monodromy} \]

\[ \text{over here, } \eta^* \text{ id} \]

Thm: (McLean, Abouzaid)

\[ \text{HW}^*(L,L) = \lim_{p} \text{HF}^*(L, \eta^p(L)). \]

\[ \text{A}_\infty \text{-modules} \]

\[ \hat{\mathcal{A}} = \text{directed Fukaya } \text{A}_\infty \text{-algebra associated to } (V_1, \ldots, V_m) \]

basis of vanishing cycles in the fibre.
Let $\mod(\hat{A})$ be the differential graded category of right $\hat{A}$-modules.

Note: $\hat{A}$ is an $\hat{A}$-module, and

\[
\xymatrix{\hat{A} \ar[r] & \hom_{\mod(\hat{A})}(\hat{A}, \hat{A})} \quad \text{asso.-quasi-iso. differential graded algebra} \quad \text{Yoneda embedding, requires unital.}
\]

Also, $\hat{A} = \bigoplus_{i=1}^{\infty} e_i \hat{A}$ respects the module structure,

and $e_j \hat{A} e_i \quad \text{quasi-iso.} \quad \hom_{\mod(\hat{A})}(e_j \hat{A}, e_j \hat{A})$.

Any $\hat{A}$-bimodule $P$ gives rise to a convolution $\text{dg}$-functor (analogue of Fourier-Mukai transform).

\[
\Phi_P : \mod(\hat{A}) \rightarrow \mod(\hat{A}) \quad M \mapsto M \otimes_{\hat{A}} P = (M \otimes_{\hat{A}} T(\hat{A}) \otimes_{\hat{A}} P).
\]

Ex: $P = \hat{A}$ diagonal bimodule, derived tensor product $\Phi_{\hat{A}} \equiv \text{Id}$ (identity functor).

Conjecture: Let $(V_1, \ldots, V_m)$ be a basis of vanishing cycles, and $(L_1, \ldots, L_m)$ the corresponding Leech tangle (pulling the basepoint to $\infty$). Then

\[
\HW^*(L_i, L_j) = \lim_{\to P} H(\hom_{\mod(\hat{A})}(e_i \hat{A} \otimes_{\hat{A}} (\hat{A}[-1]) \otimes_{\hat{A}} e_j \hat{A})).
\]
Remarks

1. Recall that $B/A \cong \hat{A}[-n]$. So the RHS can also be written as

$$\lim_{\mathbb{P}} H(\hom_{\mathbb{P}}(\hat{A}[1], \hat{A}[-1-n] \otimes_{\hat{A}} \cdots, \hat{A}[1]))$$

NB. This only depends on $A$ which is not possible, extra info lies in direct limit maps.

2. The maps forming the direct limit are induced by

$$e_i: A \cong e_i: \hat{A} \otimes_A A \leftarrow e_i: \hat{A} \otimes_A A[-1-n]$$

Rise, rise, repeat.

Hence, we need $\hat{A}$ and $[S] \in H^{n+1}(\hom_{\mathbb{P}}(\hat{A}[1], A))$ in order to do the computation.

Good: 1) everything computed is finite
   2) everything essentially computable, a priori finite amount of information

Bad: Problem of determining whether $HW = 0$ is not
   is not computable, we can use a finitely presented group to construct a space whose $HW = 0$ if group $= 0$.
   If at any point in line, stable $G \cong D$, then $0$; but not