Seidel Day II Part II.

\[ \mathcal{A} \text{ A}_\infty -\text{algebra} \]

\[ \mathcal{A} = A \otimes_R A \text{ with units added} \]

\[ \left\{ \begin{array}{l}
\mu^1_A(e_i) = 0 \\
\mu^2_A(a, e_i) = a, \quad \mu^2_A(e_i, a) = \pm a \\
\mu^d_A(--, e_i, --) = 0, \quad d > 2.
\end{array} \right. \]

\[ B \text{ A-bimodule (or unital } \mathcal{A} \text{-bimodule(e))}. \]

On the cohomology level,

\[ \mathcal{A} = H(\mathcal{A}) = R \oplus \bigoplus \text{HF}^*(V_i, V_j) \leq H(B) = B \]

and moreover \[ \mathcal{B}/\mathcal{A} \cong \mathcal{A}[-n]. \]

Additional structure: cocycle \( \Psi \in CC(A, B) \).

Notation: \( \mathcal{A} \) A_\infty algebra \( /R \), \( B \) an A-bimod. The

\[ \text{ Hochschild complex with coeff. in } B \text{ is} \]

\[ CC^*(\mathcal{A}, B) := \bigoplus_{d > 0} \text{Hom}_{R - R}(A^d, B)[-d] \]

with the differential (schematically): \( \cdots \)
\[ d\psi(-) = \eta \eta(-, \psi(-), -) = \psi(-; \eta(-), -) \]

Its cohomology is called the Hochschild cohomology \( HH(A, B) \).

The leading term of \( \eta \in \mathcal{C}(A, B) \) is \( \eta^0 \in \text{Hom}_{R^{-1}}(R, B) \)

= \( \bigoplus_i \mathbb{E}_i \mathbb{B}_i \); in our case, \( \eta^0 \in \bigoplus_i \mathcal{C}(V_i, V_i) \).

Take \( H = \mathbb{R} \times \mathbb{R}^{>0} \subset C \) (upper half-plane) and consider

\[ \mathbb{R}^\infty \times [0, 1] \subset H \] by

\[ (s, t) \mapsto -e^{\pi(s-it)} \]

Choose any one-form \( \beta \in \Omega^1(H) \) vanishing near the boundary, and equal to \( \psi(t)dt \) on \( \mathbb{R}^\infty \times [0, 1] \subset H \).

Consider:

\[ \begin{cases} 
    u : H \to M \\
    u(\delta H) = V_i \\
    (du - X \otimes \beta)^0, 1 = 0 
\end{cases} \]

\[ du - X \otimes \beta : T_2 H \to T_{u(\delta)} M. \]

On \( \mathbb{R}^\infty \times [0, 1] \), \( \beta = \psi(t)dt \Rightarrow \)

\[ (du - X \otimes \beta)^0, 1 = \frac{1}{2} ( (du - X \otimes \beta) \, ds + J(du - X \otimes \beta) \, dt ) \]

\[ = \frac{1}{2} \left( \frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X \psi(t) \right) \right) \]
This equation has
\[ \lim_{s \to \infty} u(s_t) = y_0(H) \quad y_0 \hookrightarrow \mathcal{Y}_H(V_i, \nu_V). \]

More generally, take \( S = H \setminus \{ \text{boundary points} \} \) and consider the same equation with boundary values
\[ \Rightarrow \mathcal{Y} \circ \mathcal{C}(A, B). \]
\[ i_0 < i_1 < i_2 < i_3 \]

\[ V_{i_0} = V_{i_1} = V_{i_2} = V_{i_3} \]

\( \text{NB: the equations on } H \text{ are not compatible w/ } \text{aut}(H). \text{ kind of like fixing an interior marked point. Can't divide by } \text{aut}(H)! \)

There is a canonical quasi-isomorphism
\[ \mathcal{C}(A, B) \to \text{hom}_{\text{bimod}(A)}(\hat{A}, B) \]
\[ \varphi \mapsto \hat{\varphi} \]
\[ \hat{\varphi}(\cdots, - , \cdots) = \varphi(\cdots, - , \cdots, \varphi(\cdots), \cdots) \]

\[ \text{hom}_{\text{bimod}(A)}(\hat{A}, B) = \bigcup_{r,s \geq 0} \text{hom}_{R-R}^r(A^{\otimes r} \otimes \hat{A} \circ A^{\otimes s}, B) \]

This gives rise to \( \hat{\varphi} : A \to \hat{B} \), realizing the inclusion \( H(\hat{A}) = \hat{A} \hookrightarrow B = H(B) \).
Dually, with no details, we have
\[ \psi \in \text{CC}(A, B^*) \] which gives rise to
\[ \psi \in \text{hom}_{\text{bi-mod}(A)}(B, \hat{A}) \cong (\hat{A})^* \text{ dual diagonal bimodule.} \]

The composition \[ 0 \to \hat{A} \to B \xrightarrow{\psi} A[[n]] \to 0 \] is \( \cong 0 \) (on the nose). This lifts the short exact sequence
\[ 0 \to \hat{A} \to B \to \hat{A} \to 0 \] from the previous lecture.

The sequence above has a boundary homomorphism

\[
\begin{array}{c}
S \in \text{hom}_{\text{bi-mod}}(A) \left( A[[n]], \hat{A} \right)
\end{array}
\]

\[
\xrightarrow{\text{hom}_{\text{bi-mod}}(A)} \left( \hat{A}, \hat{A} \right)
\]

depends on \( B(\ldots, \psi, \psi, \ldots) \).