Seidel Day II Part I

\((M,\omega)\) symplectic manifold, exact \((\omega = d\Theta)\),
\(c_1(M) = 0\),

\((V_1,\ldots, V_m)\) ordered collection of Lagrangian subspaces
in \(M\) (---)

\[HF^\ast(V_i, V_j) = \astl(CF^\ast(V_i, V_j), \text{differential})\]

Suppose in general position, i.e. \((V_i \cap V_j = \emptyset, \forall i \neq j,\)

no angle intersections)

For \(i < j\),
\[CF^\ast(V_i, V_j) = \bigoplus_{\text{rev.} v_i, v_j} \otimes K \times \ast\text{copy of } K\]

placed in appropriate degree.

\[\eta^1(x_3) = \sum \eta^1(x_0, x_2) x_0\]
\[\eta^1(x_0, x_2) \in K\]

counts solutions to a \(\bar{\Omega}\) equation:

\[
\begin{align*}
\{ u: \mathbb{R} \times [0, 1] & \to M \} \\
u(\mathbb{R} \times 0) & \subset V_j, \ u(\mathbb{R} \times 1) \subset V_i \\
u_{\mathbb{R} +} & \subset V_i, \ u_{\mathbb{R} -} \subset V_j \\
J(u) \frac{\partial u}{\partial s} & = 0 \quad J \text{ is a co-compact, a.c.s. (LIE: should be allowed to depend on } t)\}
\end{align*}
\]
With limits
\[ \lim_{s \to 0} u(s, \cdot) \to x_0, \quad \lim_{s \to +\infty} u(s, \cdot) = x_1. \]

\[ A = \bigoplus_{i < j} \text{CF}^*(V_i, V_j) \]

Floor–Fukaya theory makes \( A \) into an \( \text{A}_\infty \)-algebra over \( R = \mathbb{K}^m \). The structure maps are
\[ q^d : A \otimes_R \cdots \otimes_R A \to A[d-1] \quad d \geq 1 \]

R-bimodule maps

whose components are
\[ q^d_k : \text{CF}^*(V_{i_d, i_{d-1}}, V_{i_{d-1}, i_{d-2}}) \otimes \cdots \otimes \text{CF}^*(V_{i_0}, V_{i_1}) \]

\[ \to \text{CF}^*(V_{i_0}, V_{i_1})[d-1] \]

for \( i_0 \prec \cdots \prec i_d \) in \( \mathbb{N} \). The first component \( q^1 \) is just the floor differential. The higher components have the form
\[ q^d(x_0, \ldots, x_d) = \Sigma_{c \in \mathbb{C}} \nu^d(x_0, \ldots, x_d, c) x_0, \]

\( n^d \) count solutions of
\[ \sum_{u: S \to M} u(\partial^0 S) c V_i, \ldots, u(\partial^d S) c V_i, \]
\[ \bar{\delta}u = \frac{1}{2} (\delta u + J o d u o j) = 0. \]
\( S = (S_i, j) \subseteq D \setminus (0, 1) \) boundary points, together with a choice of one such point \( (S_0) \).

The limit conditions are
\[
\lim_{z \to S_i} u(z) = x_i.
\]

Note: \( S \) must be allowed to vary through its \((d-2)\)-dim. moduli space.

Modulo the usual Lie \((S \text{ must depend on } z)\), get an Aee structure.

From now on, write \( \emptyset_{d}^d = \emptyset_{d}^d \), \( CF_{\emptyset}^{\emptyset} = CF_{\emptyset}^{\emptyset} \).

Choose \( H \in \mathcal{C}^{\infty}_c(M, \mathbb{R}) \), let \( X \) be its Ham. vector field and \( \phi_H \) the time \(-1\) map of its flow.

General position: \( V_i \cap \phi_H(V_j) = \emptyset \) \( i, j \).

\[
V_i \cap V_j \cap \phi_H(V_k) = \emptyset \quad \forall \ i, j, k \text{ with } i \neq j.
\]

\[
V_i \cap \phi_d(V_j) \cap \phi_H(V_k) = \emptyset \quad \forall \ i, j, k \text{ with } j \neq k.
\]
Consider the perturbed Floer equation

\[\begin{align*}
    u : \mathbb{R} \times [0, T] &\rightarrow M \\
    u(\mathbb{R} \times 0) &\in V_i, \ u(\mathbb{R} \times 1) &\in V_j \\
    \partial s u + J(\partial_t u - \psi(t) X) &\equiv 0
\end{align*}\]

where \[\psi : [0, T] \rightarrow \mathbb{R}\] vanishes near \[T = 1\] and satisfies \[\int_0^T \psi(t) \, dt = 1\].

**Limits:**
\[\lim_{s \rightarrow +\infty} u(s, t) = y_i(t), \quad \lim_{s \rightarrow -\infty} u(s, t) = y_0(t).\]

\[\begin{align*}
    y_k(0) &\in V_i, \ y_k(1) &\in V_j, \ \frac{dy_k}{dt} &\equiv \psi(t) X
\end{align*}\]

\[X_k = y_k(1) \in \phi_{t+} (V_i) \cap V_j.\]

\[\text{CF}^\ast_B (V_i, V_j) = \bigoplus_{K \cdot x} \bigoplus_{x \in \phi_{t+} (V_i) \cap V_j} K \cdot x\]

Define a Floer differential \[N_B^1\] on it. In fact,

\[\bigoplus_{i,j} \text{CF}^\ast_B (V_i, V_j)\] is naturally an Am bimodule over \(A\).

The \(A_{op}\)-bimodule structure is given by

\[m^1_B : A \otimes_{A_{op}} B \otimes_{A} \text{CF}^\ast_B (V_i, V_j) \rightarrow \text{CF}^\ast_B (V_i, V_j),\]

\[s\]
\[ n_B \sum \frac{s_{1rr}}{B} = \sum n_B (x_0, \ldots, x_{m+1}) x_0 \]

\( n_B \) counts solutions of:

\[ \sum u : S \rightarrow M \]
\[ u(\partial_k S) < V_{i_k} \]
\[ \partial_k u + J(\partial_k u - \psi(t) X_f) = 0 \]

where \( S = (R \cup C, J) \) \{ Er point on \( R < 0 \), s pts on \( R \times 1 \) \}

\( V_{i_k} \times V_{i_k} \times V_{i_k} \times V_{i_k} \)

\( x_0 \quad S \quad x_4 \)

(\( S \) allowed to vary as always).

Codim 1 bubbling:

\[ \partial^3 = 0 \]
\[ \partial^4 + J(\partial^4 - \psi X) = 0 \]
A co-bimodule equation schematically:

\[ \mu_B(\ldots, \mu_B(\ldots, -, \ldots), \ldots) + \]

\[ \mu_B(\ldots, \mu_B(\ldots, -, \ldots), \ldots) + \]

\[ \mu_B(\ldots, \mu_A(\ldots), \ldots, \ldots, -) = 0 \]