\begin{align*}
\text{LFSCHETZ FILTRATION} & \quad \text{LEFSCHETZ FILTRATION} \\
\{ \lambda_1, \lambda_2, \lambda_3 \} & \quad \{ \lambda_1, \lambda_2, \lambda_3 \} \\
\text{SYMP. MFSD} & \quad \text{SYMP. MFSD} \\
\text{E} & \quad \text{Y} \\
\text{COMPLEMENT} & \quad \text{COMPLEMENT} \\
\end{align*}

Let \( p : E \to \mathbb{C} \) be a Lefschetz filtration with fibre \( M = \mathbb{C}^3 \) and
3 critical points. A suitable choice of vanishing paths leads to
vanishing cycles \( V_1, V_2, V_3 \subset M \).

Define \( A = \bigoplus_{i,j} \text{HF}^*(V_i, V_j) \)

This is a graded associative algebra over \( R = \mathbb{K} e_1 \oplus \mathbb{K} e_2 \oplus \mathbb{K} e_3 \).

Notation: For any \( m \geq 1 \) and any coefficient field \( \mathbb{K} \), \( R = \mathbb{K} e_1 \oplus \cdots \oplus \mathbb{K} e_n \)

is the semisimple \( \mathbb{K} \)-algebra with \( e_i^2 = e_i, x_i e_j = 0 \).

A graded associative algebra over \( R \) is a graded \( R \)-bimodule \( A \)
with an associative multiplication \( A \otimes A \to A \) a \( R \)-bimodule map.
Suppose $A$ is such an algebra. The $R$-bimodule structure decomposes

$$ A = \bigoplus_{i,j=1} e_i A e_j $$

(because by "$R$-bimodule" we mean "unital $R$-bimodule"

\[ \Rightarrow e_0 \cdots e_n \text{ acts as } 1 \text{ on } A \]

and the only nonzero component of multiplication are maps

$$ e_i A e_j \otimes e_k A e_l \longrightarrow e_x A e_y $$

(possibly non-unique)

Hence, $A$ is a graded $K$-linear category with $m$ objects.

Namely, $e_i A e_j = \begin{cases} \text{HF}^*(V_i, V_j) & \text{for } i < j \\ 0 & \text{if } i \geq j \end{cases}$

The multiplicative structure is given by triangle products.

$K$ = any field \ (if $K = \mathbb{C}$ most familiar)

For the mirror of $CP^2$

\[ \text{HF}^*(V_1, V_3) = W, \quad W \subseteq K^3 \text{ conc. in } \deg 0 \]

\[ \text{HF}^*(V_2, V_3) = W \quad \text{(char } K \neq 2) \]

\[ \text{HF}^*(V_i, V_j) = \Lambda^*(W) \]

Multiplication is $\wedge$ product.

On the side of $X = CP^2(\mathbb{K})$, consider

$$ E_1 = O_x(-1) = \Omega^2_x(2) $$

$$ E_2 = \Omega^1_x(1) \quad \Sigma \quad E = E_1 \oplus E_2 \oplus E_3 $$

$$ E_3 = O_x = \Omega^0_x(0) $$

(things from Beilinson)

Hang on... back up...

Notation. $A$ = non-unital associative algebra $/ R$

Adjoining units

$\hat{A} = R \otimes A$

extend algebra structure in obvious way.
In our case,
\[ e_i \cdot \hat{A} e_j = \begin{cases} 0 & i \neq j \\ \mathrm{HF}^* (U_i, U_j) & i < j \\ \mathbb{K} e_i & i = j \\ \mathrm{H}^* (E_i \otimes E_j) & i > j \end{cases} \]

Consider \( \bigoplus_{i,j} \mathrm{Hom}^*_X (E_i, E_j) = \mathrm{Hom}^*_X (E, \hat{E}) \)

Then: \( \mathrm{Hom}^*_X (E, \hat{E}) \cong \hat{A} \) (\( \frac{1}{2} \)HMS for \( \mathbb{C} \mathbb{P}^n \))

Remarks

1. This version depends on choices (of vanishing cycles and holomorphic vector bundles). There is a more categorical statement avoiding these choices.

2. Nevertheless, the statement is complete, since \( \mathrm{Hom}^*_X (E, \hat{E}) \) describes \( \mathbb{P}^2 (K) \) completely (so the derived category statement contains no more information).
CY hypersurfaces

\((M,\omega)\) exact symplectic \(\omega\) \(\mathbb{R}\)-valued \([\omega] \in H^2(M,\mathbb{R})\)

\(c_1(M) = 0\)

\((V_1, \ldots, V_n)\) Lagrangian spheres in \(M\) (exact, zero Maslov index, only nontrivial if \(\dim M = 1\))

\(HF^*(V_i, V_j)\) \(\mathbb{Z}\)-graded vector space/coefficients field \(\mathbb{K}\)

\(A = \bigoplus_{i,j} HF^*(V_i, V_j)\)

\(\hat{A} = \mathbb{R} \otimes A\)

\(B = \bigoplus_{i,j} HF^*(V_i, V_j)\)

Recall \(HF^*(V_i, V_j) = H^*(V_i, \mathbb{K}) \cong H^*(S^n; \mathbb{K})\) \((n = \dim M)\), and

\(HF^*(V_i, V_j) \cong HF^*(V_j, V_i)\)

Hence, as a graded vector space (graded \(\mathbb{R}\)-bimodule)

\(B \cong \hat{A} \otimes \hat{A}[-n] \quad (\hat{A}^i = (\hat{A}^*)^i)\)

\((\text{e.g. } HF^*(V_i, V_j) = \mathbb{K} e_i \otimes \mathbb{K} e_j \Rightarrow H^*(S^n; \mathbb{K}))\)

\(\deg^0 \quad \deg^m\)

Multiplicative structure of \(B\) consists of

\(A \otimes A \rightarrow \hat{A} \otimes \hat{A} \rightarrow \hat{A}\)

\(\hat{A} \otimes \hat{A} \rightarrow \hat{A}\)

\(\hat{A} \otimes \hat{A} \rightarrow \hat{A}\)

\(\text{and sym}\) \(\text{and sym}\)

\(\text{i.e. } \hat{A} \otimes \hat{A} \rightarrow \hat{A} \otimes \hat{A}\)

\(\text{i.e. } \hat{A} \otimes \hat{A} \rightarrow \hat{A} \otimes \hat{A}\)

Not a priori determined by \(A\)

\(\hat{A} \otimes \hat{A} \rightarrow \hat{A}\)

\(\hat{A} \otimes \hat{A} \rightarrow \hat{A}\)

\(\text{sym}\)

\(\text{sym}\)

So \(B\) has more information than \(A\).
Terminology:

\[ A = \text{graded assoc. algebra over } K \]

An \( A \)-bimodule \( P \) is a graded \( A \)-bimodule together with maps

\[ A \otimes_k P \xrightarrow{\mu} P \]

\[ P \otimes_k A \rightarrow P \]

satisfying the obvious conditions.

**Ex.** In the case of a unital algebra \( \hat{A} \) we want to consider unital bimodules.

**Ex.** \( P = \hat{A} \), diagonal bimodule

\[ L = \hat{A} = (\hat{A})^\vee \] dual diagonal bimodule

Given \( \hat{A} \) and an \( \hat{A} \)-bimodule \( P \), we can define the trivial extension algebra \( \hat{A} \otimes P \),

\[ (a_1, p_1) \cdot (a_2, p_2) = (a_1 a_2, a_1 p_1 + p_2 a_2) \]

(associative, unital)

Return to our geometric situation:

\[ \hat{A} = R \oplus \bigoplus_{i,j} H^j(F^* (V_i, V_j)) \]

\[ B = \bigoplus_{i,j} H^j(F^* (V_i, V_j)) \]

**Lemma:** There is a SES of \( \hat{A} \)-bimodules

\[ 0 \rightarrow \hat{A} \rightarrow B \rightarrow \hat{A}[-n] \rightarrow 0 \]

(follows from general facts about \( H^j \))
Lem: If \( \hat{A} \) is concentrated in degree 0, then \( B \) is the trivial extension.

\[
B = \hat{A} \otimes \hat{A}[\epsilon_{-1}]
\]

Pf: All unknown components of algebra structure vanish for degree reasons.

Back to \( CP^3 \):

\[
p : \mathbb{C} = (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}
\]

\[
M = p^{-1}(z) \text{ smooth fibre}
\]

\[
(V_1, V_2, V_3) \text{ in } M
\]

\[
B = \bigoplus_{i,j} \text{HF}^* (V_i, V_j) \text{ is known (trivial extension alg } \hat{A} \otimes \hat{A}[\epsilon_{-1}])
\]

\[
E, E_1, E_2, E_3 \xrightarrow{\gamma} X = p^2(K) \text{ elliptic curve}
\]

Then: \( B \cong \bigoplus_{i,j} \text{Hom}^* (\gamma^* E_i, \gamma^* E_j) \)

\[
\cong \text{Hom}^* (\gamma^* E, \gamma^* \partial) \text{ (fragment of HMS for elliptic curves)}
\]

Problem: The choice of elliptic curve is not visible; to get the full information we need to work on the cochain level (\( A_\infty \)-algebras).

On the cochain (\( A_\infty \)) level, \( B \) contains more information - which determines exactly which elliptic curve you are looking at.