Abouzaid III: Fukaya Cat. of Plumbings

\[ Q \subset M \rightarrow W(M) \rightarrow TwP(Q) \rightarrow \text{mod}(C_{x_{\Phi}}(Q, Q)) \]

Let \( M = T^*Q \), assume \( T^*Q \) has a Lefschetz fibration, \( \Phi \).

\[ Tw(A(\Phi)) \rightarrow F(M) \] (free away from Chos. 2)

Work in progress: Is an acceleration functor \( A(\Phi) \rightarrow W(M) \).

Only assess: Are structures, webs, sums everything compatible.

\[ \Rightarrow \Phi(M) \rightarrow \text{subcategory of } TwW(M) \text{ gen. by thimbles}. \]

Return to \( M = T^*Q \) with Lefschetz fibrations s.t. the thimbles are isotopic to cotangent fibres.

(More generally, \( Q \rightarrow M \) s.t. the thimbles that inherit \( L^p \) are locally fibres)

\[ \Rightarrow (F(T^*Q) \rightarrow Tw(C_{x_{\Phi}}(Q, Q))) \]
This implies that the Aoo structure on $C^*(Q)$ is equivalent to $C^*(Q)$, ordinary dy model.

Why is this non-trivial? We knew from physics that $HF^*(Q) \cong H^*(Q)$, but this is harder, requires stuff about $Q$ and $\pi_1 Q$. This may be an overkill way to determine this result.

Paul: How does this imply?

\[
\begin{align*}
A: & \quad Q \longrightarrow T_Q \\
& \quad F(T^*_Q) \longrightarrow Tw(C_\ast(\Omega_2 Q))
\end{align*}
\]

\[
\begin{align*}
\text{module over} & \quad H_\ast(\pi_3 Q) \\
\text{module over} & \quad \mathbb{Z}[\pi_3 Q] \\
\cong & \quad H_0(\Omega_2 Q). \text{ A module of rk 1 over } C_\ast(\Omega_2 Q) \\
\leftrightarrow & \quad \text{representation of } \pi_3 Q.
\end{align*}
\]

Previous result \Rightarrow rep determines the Aoo module. \text{ref. Dwyer-Greenlees-Friedlander.}
If $Q$ is simply connected, you can use "the zero section" as a generator.

$C^*(Q) \xrightarrow{\text{alg.}} K \twoheadrightarrow K$ is a $C^*(Q)$-module.

$C^*(Q)$

Implies that $C^{-\infty}(\Omega^q Q) = \text{Hom}_{C^*(Q)}(K, K)$

(This has an analogue if $Q$ is not st.; involves universal cover, more complicated).

Different strategy:

1. Prove directly that $C^F^*(Q, Q) \subseteq C^*(Q)$.
2. Observe that $HW^*(T^*_q Q, Q)$ has rank 1.

Normal alg. $\Rightarrow$ Are functor $C^W^*(T^*_q Q, T^*_q Q) \xrightarrow{\sim} C^{-\infty}(\Omega^q Q)$.

Proving this equivalence is a bit harder (like yesterday...?)

Hope: Second strategy might extend beyond cotangent bundles.

Point is we already have control on Ricci flow at $\infty$ for $T^*Q$'s, gradient flow. Might not have this?

Situation: $B, Q_1, Q_2$ smooth compact manifold, $f : B \xrightarrow{f_1} f_2$, s.t. the normal bundles are isomorphic.

Plumbing: "Glue $T^*Q_1$ and $T^*Q_2$ along a nodal of $B$"
Consider the following subset of \( \varphi \text{(plumbing)} \):

\[ CF^*(Q_1, Q_2) \]

Choose a neighborhood of \( B \) in its normal bundle \( \mu \) and embeddings \( U \hookrightarrow Q_1 \). Triangulate \( U \) and extend this triangulation to \( Q_1 \) and \( Q_2 \).

Construct another category:

\[ \text{Smith} \left[ Q_1, Q_2 \right] \]

Why is this a category? Let's see how \( \mu \) works.

\[ C^*(U) \otimes C^*(U) \rightarrow C^*(U) \]

where \( \otimes \) denotes the coproduct.
Interesting case:

\[ C^*(u) \oplus C^*(u, e^u) \to C^*(u) \]

\[ \text{cup product} \]

\[ C^*(u, e^u) \]

\[ \text{extend by 0} \]

(Our plan model is a little more complex, but works easily in simplicial story)

This is a finite-dimensional algebra.

A priori, no reason to believe that \( F(\text{anything}) \) has a \( \hat{T} \) model.

Thus: \( \text{Simp}(Q_1, Q_2) \approx \text{Fuk}(Q_1, Q_2) \)

We'll show

Idea: interpolate

\[ \text{Morse}(Q_1, Q_2) \]

\[ \text{interpolate} \]

\[ \text{This part: skip other part.} \]

Pick \( h_1, h_2, h_{12}, h_{21} \)

\[ \text{Morse on } Q_1 \]

\[ \text{Morse on } Q_2 \]

\[ h_{12} \ (h_{21} \text{ has sources/destinations pointing } \)

\[ C^*(h_{12}) \] \text{grad. flow near boundary}

\[ (\text{Morse}(h_{12}) \) w. A.

\[ \text{Ruan} \]

\[ C^*(h_{21}) \]

\[ \text{As stated: Count periodic gat. issues in the non-Hausdorff manifold obtained by giving } Q_1 \text{ a 2-neighborhood.} \]
What is perturbed?

How to define product? \( (\text{Hom}(h_1) \otimes \text{CM}^*(h_i)) \rightarrow \text{CM}^*(G) \)

Fix \( h_1 \).

- \( \partial h_1 \)

- \( \partial h_1 \)

- \( \partial h_1 \)

non-transverse!

In Floer theory, we allow

Similarly, in our case, allow

Arbitrary (domain-dependent) perturbations

- \( \partial h_1 \)

- \( \partial h_1 \)

\( \partial h_1 \)

Extend to a consistent perturbation on the moduli space of gradient trees.

How to show \( \text{Maslov}(Q_1, Q_2) = \delta(Q_1, Q_2) \)?

Previous attempts use degeneration techniques—scary and hard. Attempt to make \( = \) on nose, but only need Ann equivalence.
Idea: Use the Log'\( n \) (cotangent fibre) function on \( T^*Q \) to construct an \( A_{\infty} \) functor.

extends to plumbing

\[ Q_2 \]

Note there's a choice in this situation. Do

\[ Q_1 \]

Instability corresponds to asymmetry in \( Q_0 \).

Grey line identifies \( w_6 \) or \( Q_1 \) w.r.t. \( T^*Q_1 \).

\[ Q_2 \rightarrow \text{a solution of } T^*Q_1 \text{, defined over image of } U. \]

Graph of \( dh \)

\[ h: U \rightarrow R \]

either \( h_{12} \) or \( h_{21} \)

depend on choices.

Now how to construct this functor?

\[ \text{graph}(dh) \]

Initial \[ CF^*(Q_1, Q_1) \]
defined using

Hamilton perturbation given by \( h_2 \),

\[ \text{gen of } CF^* = \text{gen of } CM^* \]

\[ h \rightarrow h \times h \]
Conj: $HH$ (this alg)

"broken loops"

Disc w/ 3 marked pts

film.

$\text{grad}(\theta)$

$D^2$

$([-\infty, 0], \mathbb{C})$

$\mathbb{C}^n$

complicated moduli space

to interpolate between

$A_{\infty}$ structures.

Relation to ref. SFT:

\[ \text{conj: } HH^*(T^*Q_1 \times T^*Q_2) \simeq HH^*(\text{Fuk}(Q_1, Q_2)) \]

$B = \mathfrak{p}^+.$

The bar complex which compiles $HH^*(\text{Sym}(Q_1, Q_2))$

"looks like" the space of broken loops on $Q_1, B, Q_2$

If you try to do this geometrically, difficult

Involves a string bracket with the chain (algebraic)

encoded algebraically in the Hochschild complex.

\[ \text{Alg}_{B^}\]

\[ C = (\mathcal{C}) \text{ encoding Hochschild complex.} \]