Derived Category of Coherent Sheaves

1. The Derived Category

I. Abelian categories

Primary examples:
- $R$-Mod $R$-modules
- $O_X$-Mod sheaves on a ringed space $(X, O_X)$
- $O_{cohX}$ or $Coh_X$ of (quasi) coherent sheaves on $X$.

Defn: An additive category $A$ is a category enriched over $Ab$.
(i.e. The Hom-sets of $A$ are abelian groups and composition is bilinear)
and which possesses biproducts, i.e. for each $X, Y \in \text{Ob} A$ there is
another object $X \oplus Y$ which is a product and coproduct for $X, Y$.

Let $A$ be an additive category, let $\varphi : X \to Y$ be a morphism in $A$.
$K \xrightarrow{i} X$ is called a kernel of $\varphi$ if $\varphi \circ i = 0$ and $i$ is
universal w.r.t. the property:

\[ K \xrightarrow{i} X \xrightarrow{\varphi} Y \xrightarrow{0} \]

A morphism $\pi : Y \to C$ is a cokernel of $\varphi$ is $\pi \circ \varphi = 0$ and $\pi$
is universal w.r.t. the property:

\[ X \xrightarrow{\varphi} Y \xrightarrow{\pi} C \]

Let us assume $A$ has kernels and cokernels.

Consider

\[ \xymatrix{ K & C \ar[l] \ar[r]^i & X \ar[d] \ar[r]^\varphi & Y \ar[d] \ar[r]^\pi & C \ar[d] \ar[r] & \} \]

$\text{im } \varphi = I \xrightarrow{\exists ! q} I' \xrightarrow{\text{im } \varphi}$ but this map need not be an iso.

by universal property.
Defn: An abelian category $A$ is an additive category with kernels and cokernels and coin $\circ \subseteq \text{im} f \circ \subseteq \text{im} f$ in $A$.

Rmk: Each small abelian cat can be embedded fully, faithfully and exactly in a category of $R$-modules.

Let $A$ be an abelian category. Let

$$X \overset{\phi}{\rightarrow} Y \overset{\psi}{\rightarrow} Z$$

be a sequence in $A$ with $\psi \circ \phi = 0$. Then

$$\text{im} \phi \rightarrow \ker \psi$$

so we can define $H = \ker \psi / \text{im} \phi$

the homology of the sequence at $Y$.

If $H = 0$ we call the sequence exact.

E.g. Let $F : A \rightarrow B$ be an additive functor between abelian categories.

Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence in $A$.

Then $0 \rightarrow FX \rightarrow FY \rightarrow FZ \rightarrow 0$ need not be exact.

If it is then $F$ is called exact.

E.g.

$$0 \rightarrow \mathbb{Z} \overset{3}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

Not exact.

Defn: If $I \in A$ is called injective if $\text{Hom}(\text{___}, I) : A \rightarrow \text{Ab}$ is exact.

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact,

$$0 \rightarrow \text{Hom}(C, I) \rightarrow \text{Hom}(B, I) \rightarrow \text{Hom}(A, I)$$

is always exact (left exact functor).

So $I$ injective $\Leftrightarrow A \overset{\epsilon}{\rightarrow} B$

$$\downarrow$$

I

\text{exact.}
The Derived Category of an Abelian Category

To compute $\text{Ext}^i(X,A)$ one first replaces $A$ by an injective resolution:

\[ A \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots \]

with $I^i$ being injective.

Apply $\text{Hom}(X,-)$ to this resolution:

\[ 0 \rightarrow \text{Hom}(X,I^0) \rightarrow \text{Hom}(X,I^1) \rightarrow \ldots \]

Then $\text{Ext}^i(X,A) \cong H^i\left(\text{Hom}(X,I^i)\right)$.

Diagram chase gives you the LES

\[ 0 \rightarrow \text{Ext}^0(X,I) \rightarrow \text{Ext}^0(B,I) \rightarrow \text{Ext}^0(A,I) \rightarrow \text{Ext}^1(C,I) \rightarrow \ldots \]

Idea: identify $A$ with all its resolutions.

We should apply a functor only to the right resolution.

**Def:** A complex in $A$ is a

**Def:** Chain complex, morphisms = chain maps. This is called $\text{Kom}(A)$.

**Rem:** Assume $A$ possesses all small products.

\[ \text{Hom}^k(X,Y) = \left\{ f^k : X^k \rightarrow Y^k \mid (\epsilon^k) \right\} \]

\[ d^k : \text{Hom}^k \rightarrow \text{Hom}^{k+1} \]

\[ (f^k) \mapsto d^{k+1} f^k = (-1)^k f^k \circ d^k \]

This is a complex.

Note that $\text{Hom}(X,Y) = \ker\left( \text{Hom}^0 (\epsilon^0) \right)$.

Note that $\text{im}(d^n) = \text{chain homotopies}$.

We call the homotopy category $\mathcal{K}(A)$:

\[ \text{Ob} \mathcal{K}(A) = \text{Ob} \text{Kom}(A) \]

\[ \text{Mor} \mathcal{K}(A) = H^0 \text{Hom}^1 \left( X^1, Y^1 \right) \]

\[ = \text{Mor} \mathcal{K}(A) / \text{homotopy} \]
The cohomology functors $H^*: K(A) \rightarrow \text{Ab}$ are still well-defined.

Consider again $A$ with its injective resolution $I^*$. This can be viewed as a morphism of complexes

$$\cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow Q \rightarrow \cdots$$

$$\cdots \rightarrow 0 \rightarrow I_n \rightarrow I_{n-1} \rightarrow I_{n-2} \rightarrow \cdots$$

This is a homotopy equivalence ($\Delta$-functor, have same homology).

$\Rightarrow$ a quasi-isomorphism.

Idea: formally invert all quasi-isomorphisms (like localizing a ring).

Def: The derived category of $A$ is the category $K^*(A)$ localized at the quasi-isomorphism, i.e., a functor $K^*(A) \rightarrow D^+(A)$ that maps qis to isomorphisms and is universal with this property.

Rank: $\text{Ob } D^+(A) = \text{Ob } K^*(A)$

$$\text{Hom}_{K^*(A)}(X^*, Y^*) = \left\{ \begin{array}{ll}
X^* & \text{if } X^* \sim Y^*
Y^* & \text{if } Y^* \sim X^*
\end{array} \right.$$
III. Structure of Derived Category.

Facts: There are autoequivalences $D(A) \to D(A)$ shift functors.

In particular, can look at $\text{Hom}_A(X, Y[-i])$ for $X, Y \in \text{Ob} A$.

\[ \cdots \to X \to Y \to \cdots = \text{Ext}^i(X, Y) \]

\[ \cdots \to 0 \to X \to Y \to 0 \to \cdots \]

Remark: Let $I \subset A$ be the full subcategory of injective objects in $\text{A}$.

$q_i$ in $K^+(I)$ is already a homotopy equivalence.

\[ K^+(I) \cong D^+(I) \] is an equivalence of categories.

If $A$ has enough injectives, $K^+(I) \to D^+(I) \to D^+(A)$

is an equivalence of categories.

**Defn.** A triangle in $D(A)$ is a diagram of the form $K \to L \to M \to K[1]$.

The triangle is distinguished if it is isomorphic to a triangle of the form $K' \to K' \oplus K'[1] \oplus L' \to K'[1] \oplus L'[1] \to K'[1]$.

for a morphism $f : K \to L'$.

**Prop.** $0 \to A \to B \to C \to 0$ is exact in $K_0(A)$, if and only if it is an exact sequence $0 \to A \xrightarrow{f} \text{Cyl}(q) \xrightarrow{g} \text{Cyl}(f)$.

**Rem.** Given a distinguished triangle $K \to L \to M \to K[1]$, it induces a LES

\[ H^i(K) \to H^i(L) \to H^i(M) \to H^{i+1}(K) \to \cdots \]
Def: A triangulated category $A$ is an additive category with a shift and a class of distinguished triangles $T$ s.t. axioms.

[Non-English content]

Derived functors

Let $F: A \to B$ be a left-exact functor between abelian categories. Can we extend it to $F: \mathcal{D}(A) \to \mathcal{D}(B)$? $0 \to A \to B \to C \to 0$ exact

$\Rightarrow \exists 0 \rightarrowtail z \in \mathcal{D}(A) \Leftarrow F(0 \to A \to B \to C \to 0) \cong 0$ but $0 \to F(A) \to F(B) \to F(C) \to 0 \neq 0$

$\Rightarrow$ be more careful.

Assume $3$-class of objects $F$ adapted to $F$ (stable under finite direct sums, every object in $A$ is a subobject of an object in $B$, and $F$ maps acyclic complexes in $A$ to acyclic ones in $B$ (i.e., sends exact sequences $\rightarrowtail$ exact sequences))

Then, we can define $RF: \mathcal{D}^+(A) \to \mathcal{D}^+(B)$ as follows:

Given $A \in \mathcal{D}^+(A)$, replace $A$ by a q.i. complex in $B$, say $A'$.

Apply $F$ termwise: $RF(A') = F(A')$.

Prop: $RF: \mathcal{D}^+(A) \to \mathcal{D}^+(B)$ sends exact triangles to exact triangles.

1) $RF$ is the best approximation for $F$.

2) $RF$ is the best approximation for $F$.

[Diagram]

1) Means that $0 \to A \to B \to C \to 0$ is mapped to a distinguished triangle $RF(A) \to RF(B) \rightarrowtail M \to RF(C)$.
Taking cohomology:

\[ R^i F(A) \rightarrow R^i F(B) \rightarrow R^i F(C) \rightarrow R^{i+1} F(A) \rightarrow \cdots \]

E.g. \( R^i \text{Hom}(X,-) = \text{Ext}^i(X,-) \).

**Remark:** Derived \( \Rightarrow \) left exact functors.

**Spectral sequences:**

E.g. \( X \rightarrow Y \rightarrow Z \)

\[ \Rightarrow F: \text{Coh}(X) \rightarrow \text{Coh}(Y) \text{ pushforward} \]

\[ C: \text{Coh}(Y) \rightarrow \text{Coh}(Z) \]

\[ R(G \circ F) = RG \circ RF \]

\( \text{homology of a double complex} \Rightarrow \text{spectral sequence} \).

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6. **Applications of the Derived Category**

I. Grothendieck–Verdier Duality

**Recall:** Serre duality

\( X \) projective Cohen–Macaulay scheme of equidimensional \( n \) over an alg. closed field \( k \). Then \( \mathcal{F} \) a dualising sheaf \( \mathcal{O}_X^* \) on \( X \) s.t. there are natural isomorphisms

\[ \text{Ext}^{n-i}(\mathcal{F}, \mathcal{O}_X) \cong H^i(X, \mathcal{F}) \text{ for any coherent sheaf } \mathcal{F} \]

\[ H^{n-i}(X, \mathcal{O}_X \otimes \mathcal{F}^*) \text{ for } \mathcal{F} \text{ a-bundle, locally free.} \]

**Theorem:** \( f: X \rightarrow Y \) proper morphism of noetherian separated schemes. Then:

**bullshit.**

II. \( X \) smooth proj. variety \( \Rightarrow D^b(X) = D^b(\text{Coh} X) \).

**Fourier–Mukai transforms:**

\[ \Phi \in D^b(X) \quad \mathcal{E} \in D^b(X \times Y) \]

\[ \begin{array}{c}
\Phi \\
\downarrow \\
X \\
\downarrow \\
Y
\end{array} \]

\[ \begin{array}{c}
\mathcal{E} \\
\downarrow \\
X \times Y
\end{array} \]
$$\phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$$ is called a FM transform.

Then (Orlov): Let $F : D^b(X) \to D^b(Y)$ be an equivalence of derived categories. Then there is a unique $\phi_{\mathcal{E}} \in D^b(X \times Y)$ such that $F = \phi_{\mathcal{E}}$.

Cor: $D^bX \cong D^bY \Rightarrow \dim X = \dim Y$.

Then (Bondal, Orlov): Let $X$ have ample canonical bundle. Then $D^bX \cong D^bY \Rightarrow X \cong Y$.

III. The Derived Category of coherent sheaves on $\mathbb{P}^n$.

Let $N = H^0(\mathbb{P}^n, \mathcal{O}(1)), \quad \mathbb{P}^n = \mathbb{P}(V)$.

Euler sequence:

$$0 \to \Omega \to V \otimes \mathcal{O}(1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

$$0 \to \Omega(-1) \to V \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

Define a homomorphism

$$p^* \mathcal{O}(-1) \otimes q^* \Omega(1) \xrightarrow{s} \mathcal{O} \otimes \mathbb{P}^n \otimes \mathbb{P}^n$$

$s(H, H') : (y, u') \mapsto g(u')$$

$$e \in \mathcal{H}^* \quad \text{vanishes exactly when } u' \in H$$

Im(s) is the ideal sheaf of $\Delta c \mathbb{P}^n \times \mathbb{P}^n$.

$$0 \to \Lambda^m(p^* \mathcal{O}(-1) \otimes q^* \Omega(1)) \to \cdots \to \Lambda^1(p^* \mathcal{O}(-1) \otimes q^* \Omega(1)) \otimes \mathcal{O}_{\Delta c \mathbb{P}^n} \to 0$$

$$\cong p^* \Omega(-1) \otimes \Omega^m(-m)$$

is an exact sequence on $\mathbb{P}^n \times \mathbb{P}^n$. 
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- $\text{Coh}_X$ or $\text{Coh}_X$ of (quasi) coherent sheaves on $X$.

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(i.e. The Hom-sets of $\mathcal{A}$ are abelian groups and composition is bilinear)
and which possesses biproducts, i.e. for each $X, Y \in \text{Ob} \mathcal{A}$ there is
another object $X \oplus Y$ which is a product and coproduct for $X, Y$.

Let $\mathcal{A}$ be an additive category, let $\varphi: X \rightarrow Y$ be a morphism in $\mathcal{A}$.
$\begin{array}{ccc}
K & \xrightarrow{\pi} & X
\end{array}$ is called a kernel of $\varphi$ if $\varphi \circ \pi = 0$ and $\pi$
is universal w.r.t. the property:

$\begin{array}{ccc}
K & \xrightarrow{\pi} & X & \xrightarrow{\varphi} & Y \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0
\end{array}$

A morphism $\begin{array}{ccc}
Y & \xrightarrow{\pi} & C
\end{array}$ is a cokernel of $\varphi$ is $\varphi \circ \pi = 0$ and $\pi$
is universal w.r.t. the property

$\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y & \xrightarrow{\pi} & C \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0
\end{array}$

Let us assume $\mathcal{A}$ has kernels and cokernels.

Consider

$\begin{array}{ccc}
\delta \circ & : & K \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
I = \ker (K \rightarrow X) & \xrightarrow{\sim} & X/\ker \varphi \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
\Gamma' = \ker (Y \rightarrow C) & \xrightarrow{\sim} & \text{im} \varphi \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
\text{coim} \varphi = I & \longrightarrow & I/\text{im} \varphi
\end{array}$

but this map need not be an iso.

by universal property.
Defn: An abelian category $\mathcal{A}$ is an additive category with kernels and cokernels and $\text{cok}(f) \subseteq \text{im}(f)$ for $f: X \to Y$ in $\mathcal{A}$.

Rmk: Each small abelian cat. can be embedded fully, faithfully and exactly in a category of $R$-modules.

\[
\begin{align*}
X \xrightarrow{\phi} Y \xrightarrow{\psi} Z
\end{align*}
\]
be a sequence in $\mathcal{A}$ with $\text{cok}(\psi) = 0$. Then

\[
\text{im}(\phi) \to \ker(\psi)
\]
so we can define $H = \ker(\psi)/\text{im}(\phi)$

the homology of the sequence at $Y$.

If $H = 0$ we call the sequence exact.

E.g. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

\[
\begin{align*}
0 \to X \to Y \to Z \to 0
\end{align*}
\]
be an exact sequence in $\mathcal{A}$.

Then $0 \to FX \to FY \to FZ \to 0$ need not be exact.

If it is then $F$ is called exact.

E.g. $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$

is exact.

Defn: $I \in \mathcal{A}$ is called injective if $\text{Hom}(\mathcal{A}, I): \mathcal{A} \to \mathcal{A}$ is exact.

If $0 \to A \to B \to C \to 0$ exact,

\[
0 \to \text{Hom}(C, I) \to \text{Hom}(B, I) \to \text{Hom}(A, I)
\]

is always exact (left exact functor).

So $I$ injective $\iff$ $A \to B$

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \phi \\
I \end{array}
\]

is exact.
II. The Derived Category of an Abelian Category

To compute $\text{Ext}^i(X, A)$ one first replaces $A$ by an injective resolution:

$$A \to I^0 \to I^1 \to \cdots$$

with $I^i$ being injective.

Apply $\text{Hom}(X, -)$ to this resolution:

$$0 \to \text{Hom}(X, I^0) \to \text{Hom}(X, I^1) \to \cdots$$

Then $\text{Ext}^i(X, A) = H^i(\text{Hom}(X, I^i))$.

Diagram chase to get the LES

$$0 \to \text{Ext}^i(X, A) \to \text{Ext}^i(Y, A) \to \text{Ext}^i(B, A) \to \text{Ext}^i(\mathbb{C}, A) \to \cdots$$

Ideas: Identify $A$ with all its resolutions.

One should apply a functor only to the right resolution.

Defn: A complex in $\mathbb{A}$ is a

Defn: Chain complex, morphisms = chain map. This is called $\text{Kom}(\mathbb{A})$.

Remk: (Assume $\mathbb{A}$ possesses all small products)

$$\text{Hom}^k(X', Y') = \{ f^i: X^i \to Y^{i+k} | i \in \mathbb{Z} \}$$

$$d^k: \text{Hom}^k \to \text{Hom}^{k+1}$$

$$(f^i) \mapsto (d^{i+k} \cdot f^i - (-1)^i f^{i+1} \cdot d_i^k)$$

This is a complex.

Note that $\text{Hom}(X', Y') = \ker(\text{Hom} \circ d^k)$.

Note that $\text{im}(d^{i+1}) = \text{chain homotopies}$.

We call the homotopic category $\mathcal{K}(\mathbb{A})$:

$$\text{Ob} \: \mathcal{K}(\mathbb{A}) = \text{Ob} \: \text{Kom}(\mathbb{A})$$

$$\text{Mor} \: \mathcal{K}(\mathbb{A}) = \frac{\text{Mor} \: \text{Kom}(\mathbb{A})}{\text{homotopy}}$$
The cohomology functors $H^i : K(A) \to A^i$ are still well-defined.

Consider again $A$ with its injective resolution $I^*$. This can be viewed as a morphism of complexes

$$\cdots \to 0 \to A \to 0 \to 0 \to \cdots$$

$$\cdots \to 0 \to I_0 \to I_1 \to I_2 \to \cdots$$

This is a homotopy equivalence $(k \Rightarrow$ have same homology).

Def: The derived category of $A$ is the category $K(A)$ localized at the quasi-isomorphism, i.e. a functor $K(A) \to D(A)$ that maps qis to isomorphisms and is universal with this property.

**Rand:** $D(A) = \text{Qis} K(A)$

$\text{Hom}_{D(A)}(X', Y') = \left\{ \begin{array}{c} \text{qis} \end{array} \right\}$

Things that are true: $X', Y' \Rightarrow$ we can make a single roof $X$.

(Only for $K(A)$, not $\text{Hom}(A)$).

You can find a common denominator so it is possible to add $h \circ k^{-1} + g \circ f^{-1}$.

i.e. $3, 2$ s.t. $3^{-1} h \circ k \circ 3^{-1} = 5 \circ p^{-1}$...

\[ \begin{array}{c} h \circ k + g \circ f^{-1} = (q + t) \circ p^{-1}. \end{array} \]
III. Structures of Derived Category

E.g. There are autoequivalences $D^+(A) \cong D^+(A)$ shift functors.

In particular we can look at $\text{Hom}_D(X, Y[i])$ for $X, Y \in \text{Ob } A$.

\[ \cdots \longrightarrow i^0 \longrightarrow i^1 \longrightarrow \cdots \]
\[ \cdots \longrightarrow 0 \longrightarrow i \longrightarrow 0 \longrightarrow i \longrightarrow \cdots \]
\[ \text{Ext}^i(X, Y) \]

Remark: Let $I \subseteq A$ be the full subcategory of injective objects in $A$.

$q_! : \text{in } K^+(I) \text{ is already a homotopy equivalence}$

Since $q_!$ is exact all terms $q_!(X)$ are injective.

\[ K^+(I) \cong D^+(I) \] is an equivalence of categories.

If $A$ has enough injectives, $K^+(I) \rightarrow D^+(I) \rightarrow D^+(A)$

is an equivalence of categories.

Def: A triangle in $D(A)$ is a diagram of the form $K \rightarrow L \rightarrow M \rightarrow K[1]$.

The triangle is distinguished if it is isomorphic to a triangle of the form

\[ \begin{align*}
&K' \rightarrow K' \oplus K'[1] \oplus L' \rightarrow K[1] \oplus L[1] \rightarrow K[1]'.
\end{align*} \]

for a morphism $f : K' \rightarrow L'$.

Prop: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact in $\text{Kom}_D(A)$, it is isomorphic to an exact sequence $A \rightarrow A \rightarrow \text{Cyl}(f) \rightarrow \text{Cone}(f)$.

Remark: Given a distinguished triangle $K \rightarrow L \rightarrow M \rightarrow K[1]$ it induces a LES

\[ H^i(K) \rightarrow H^i(L) \rightarrow H^i(M) \rightarrow H^{i+1}(K) \rightarrow \cdots \]
Def: A triangulated category $\mathcal{A}$ is an additive category with a shift and a class of distinguished triangles $T$ s.t. exact

Derived functors

Let $F: A \to B$ be a left-exact functor between abelian categories. Can we extend it to $F: \text{D}^+(A) \to \text{D}^+(B)$? $\quad 0 \to A \to B \to C \to 0$ exact
$\Rightarrow$ $0$ in $\text{D}(A) \Rightarrow F(0 \to A \to B \to C \to 0) = 0$ but $0 \to F(A) \to F(B) \to F(C) \to 0$ $\neq 0$

$\Rightarrow$ be more careful.

Assume 3. class of objects $I$ adapted to $F$ (Stable under finite direct sums, every object in $A$ is a subobject of an object in $I$, and s.t. $F$ maps acyclic complexes in $I$ to acyclic ones in $B$ (i.e. sends exact sequences $\to$ exact sequences))

Then we can define $RF: \text{D}^+(A) \to \text{D}^+(B)$ as follows:

Given $A \in \text{D}^+(A)$, replace $A$ by a quasi complex in $I$, say $A'$.

Apply $F$ termwise: $RF(A) = F(A')$.

Rmk 1) $RF: \text{D}^+(A) \to \text{D}^+(B)$ sends exact triangles to exact triangles.

2) $RF$ is the best approximation for $F$

\[
\begin{array}{ccc}
A & \to & \text{D}^+(A) \\
F & & \downarrow \text{RF} \\
B & \to & \text{D}^+(B).
\end{array}
\]

natural trans.

1 means that $0 \to A \to B \to C \to 0$ is mapped to a distinguished triangle $RF'(A) \to RF'(B)$

$\to RF'(C)$.
Taking cohomology:

\[ R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \cdots \]

E.g. \( R^i \text{Hom}(X, -) = \text{Ext}^i(X, -) \).

**R mk**: Dualiz \( \Rightarrow \) left exact functors.

**Spectral sequences**:

E.g. \( X \to Y \to Z \)

\[ \Rightarrow F_\ast : \text{Coh}(X) \to \text{Coh}(Y) \text{ pushforward} \]

\[ C_\ast : \text{Coh}(Y) \to \text{Coh}(Z) \]

\[ R(G \circ F) = RG \circ RF \]

Homology of a double complex \( \Rightarrow \) spectral sequence.

(II) Applications of the Derived Category

I. Grothendieck–Verdier Duality

**Recall**: Serre duality

\( X \) projective Cohen–Macaulay scheme of equidim \( n \) over an alg. closed field \( k \). Then \( \mathcal{F} \) a dualising sheaf \( \omega_X^n \) on \( X \) s.t. there are natural isomorphisms

\[ \text{Ext}^{n-i}(\mathcal{F}, \omega_X^n) \cong H_i(X, \mathcal{F}) \text{ for any coherent sheaf } \mathcal{F}. \]

\[ H^{n-i}(X, \omega_X^n \otimes \mathcal{F}) \text{ for } \mathcal{F} \text{ coherent, locally free}. \]

**Then**: \( f : X \to Y \) proper morphism of Noetherian separated schemes. Then

**bullshit**.

II. X smooth proj. variety \( \Rightarrow D^b(X) = D^b(\text{coh } X) \).

Fourier–Mukai transforms:

\[ X \times Y \]

\[ \xi \in D^b(X) \]

\[ \xi \in D^b(X \times Y) \]

\[ \xi \]

\[ Y \]
$\rho_{\Phi} E : q^* F \hookrightarrow \rho_* (\Phi) F$

**Defn:** $\Phi : D^b(X) \to D^b(Y)$ is called a FM transform.

**Theorem (Orlov):** Let $F : D^b(X) \to D^b(Y)$ be an equivalence of derived categories. Then there is a unique $\Phi \in D^b(X \times Y)$ such that $F = \Phi_{\rho^*}$.

**Cor:** $D^b X \cong D^b Y \iff \dim X = \dim Y$.

**Theorem (Bondal, Orlov):** Let $X$ have ample canonical bundle. Then $D^b X \cong D^b Y \implies X \cong Y$.

### III. The Derived Category of Coherent Sheaves on $\mathbb{P}^n$.

Let $V = H^0(\mathbb{P}^n, \mathcal{O}(1))$, $\mathbb{P}^n = \mathbb{P}(V)$.

**Euler sequence:**

$0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$

$0 \to \mathcal{O}(n) \to V \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$

Define a homomorphism

$p^* \mathcal{O}(-1) \otimes q^* \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$

$s(H, H') : (p^*, q^*) \to (p(u), q(v))$

$\epsilon(V)^* \to H'$

 VANISHES EXACTLY WHEN $V \in H$?

$\text{im}(s)$ is the ideal sheaf of $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

$0 \to \Lambda^m (p^* \mathcal{O}(-1) \otimes q^* \Omega(1)) \to \Lambda^m (p^* \mathcal{O}(-1) \otimes q^* \Omega(1)) \otimes \mathcal{O} \to 0$

$0 \to p^* \mathcal{O}(-m) \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$

is an exact sequence on $\mathbb{P}^n \times \mathbb{P}^n$. 


Let $F$ be an object in $D^b(\mathbb{P}^n \times \mathbb{P}^n)$ of the form

$$O_a \otimes q^* F \quad F \in D^b(\mathbb{P}^n)$$

$H^0(F)$ lies in a subcategory generated by sheaves of the form

$$p^* O(-i) \otimes q^* F.$$

Apply $p_+(\cdot) \mapsto H^0(F)$ is generated by objects of the form $\mathcal{O}(i)\otimes \mathcal{R} \Gamma(X, F)$

$$\text{vector space}$$

$$\Rightarrow \quad O(0), \ldots, O(-n) \text{ generate } D^b(\mathbb{P}^n) \text{ as a triangulated category.}$$

*Thm. (Beilinson)*: Let $A'$ be a graded algebra. Let $A' \langle i \rangle \ldots \langle i \rangle$ be the free $A'$-module with one generator of degree $i$. Let $M_{[0, n]}(A')_{\mathbb{Z}}$ be the category of graded $A'$-modules isomorphic to direct sums of the form $A' \langle i_1 \rangle \oplus \ldots \oplus A' \langle i_n \rangle \quad 0 \leq i \leq n$

Let $K^b_{[0, n]}(A') := K^b(M_{[0, n]}(A'))$

Write $K_A := K^b_{[0, n]}(\Lambda V^*)$, $K_S := K^b_{[0, n]}(S \cdot V)$ (triangulated categories)

Let $F_1 : K_A \longrightarrow D^b(\mathbb{P}^n)$, $\Lambda V^* \langle i \rangle \mapsto \mathcal{O}(i)$

Let $F_2 : K_S \longrightarrow D^b(\mathbb{P}^n)$, $S V \langle i \rangle \mapsto \mathcal{O}(i)$

are equivalences of derived categories.