$M = C^2 - \{ \text{smooth fibres} \} = C^2 - \pi^{-1}(0,0)$

smooth fibre is $C^2 - \{ 0 \}$ + points

$M$ a smooth affine variety $\Rightarrow$ can be holomorphically embedded in $C^n$

$i: M \to C^n$

$E_M = i^* E_{C^n}$

$[1:0:0], [0:1:0], [0:0:1], [1:1:1]$ are the points where $\pi$ is not defined, and they are contained in $\pi^{-1}(0)$.

singular fibres are 2 $C^*$s intersecting transversely - exactly these:

\[\begin{array}{cccc}
\text{locally around a singular point } & & & \\
\Pi = z_1 z_2 = 1/2 \left( (z_1 + z_2)^2 + (i z_1 - i z_2)^2 \right) & & & \\
\end{array}\]

$\Pi: E \to S$. Away from $E_{\text{sing}}$ we have a connection: a 2-plane field $H$ which is the $\omega$-orthogonal plane to the vertical tangent spaces $T_p E = T_p F \oplus H_p$.

An embedded path $f: [0,1] \to S$ gives us a symplectomorphism $\Gamma_f : E_{f(1)} \to E_{f(0)}$
Vanishing Cycles

Choose paths from each punctual point in \( \pi(E^\text{sing}) \) to some point not in \( \pi(E^\text{sing}) \). We get a vanishing thimble \( T = \pi^{-1}(f_0) \cap \pi^{-1}(f) \) which parallel transport along \( f \) to the singular point in \( E_f(0) \), along with the singular point in \( E_f(0) \).

Picture:

\[ T \rightarrow E \]

\[ \pi \]

\[ S \]

Thm: \( T \) is a Lagrangian subman if \( E \) diffeomorphic to a ball whose boundary is a Lagrangian sphere in \( E_f(0) \) - (the vanishing cycle).

Monodromy: We can recover \( E \) (up to symplectic deformation through Lefschetz filtrations) from the vanishing cycles.

Parallel transport map \( \Gamma_g : E_f(0) \rightarrow E_f(0) \)

This should be a symplectomorphism defined up to Hamiltonian isotopy.

Thm: The monodromy around \( g \) is isotopic to a Dehn twist around \( V \) (the vanishing cycle of \( f \)).
What is a Dehn twist?

A symplectomorphism that is non-trivial only in a neighborhood of $V$ (which must be $\cong D_\delta^* V$ = cotangent vectors of length $< \delta$, using spherical metric on $V$).

Let $Y: D_\delta^* V \to \mathbb{R}$, $\tau(a) = \text{length of } a$.

$$H := h(\tau)$$

The flow of $X_H$ away from $\tau = 0$ is well-defined (but note $h$ not $C^1$ at $0 = \text{not defined there}$).

Define $\tau: D_\delta^* V \to D_\delta^* V$ as $\text{Flow}_{X_H}$ away from $0$ and the antipodal map at $\tau = 0$ ($\pi \mapsto -\pi$).

E.g. $T^* S^1$.

$$\omega = d\phi \wedge d\theta$$

Using this theorem, we can construct $E$ (up to symplectic isotopy through Lifschitz filtrations) from a smooth symplectic mfd $F$ (the fibre) and an ordered collection of smooth Lagrangian embedded $\phi: S^n \rightarrow F$. 
Fukaya Categories

$M$ symplectic with bdy $\partial M \subset \Omega = \partial \nu$, $\nu|_{\partial M}$ = contact form, convexity condition.

$C_1(M) = 0$

Donaldson category

Objects: exact compact lagrangians $L \in \mathcal{L}^{2n}$, $\Theta|_L = df$

with a grading $\mathcal{D}_L$ and a spin structure.

Morphisms: $\text{Hom}(L_1, L_2) := \text{HF}(L_1, L_2)$

$\bullet$ Choose a Hamiltonian $H$ s.t. $L_1$, $\Phi^h_t(L_1)$ intersect transversely.

$\Rightarrow |L_1 \cap \Phi^h_t(L_1)| < \infty$.

$\bullet$ Choose an a.e. struct compatible with $\omega$.

Each elt $x \in L_1 \cap \Phi^h_t(L_2)$ has an index $|x|$.

$CF_k = \bigoplus_{x \in L_1 \cap \Phi^h_t(L_2) \text{ with index } k} \mathbb{Z}$

$d: CF_k \rightarrow CF_{k+1}$

$d(x) = \sum_{y \in L_1 \cap \Phi^h_t(L_2)} \# (\mathcal{M}(y, x)_{\mathbb{R}}) y$

$\mathcal{M}(y, x) = \text{maps } u: \mathbb{R} \times [0, 1] \rightarrow M$

$u(s, 0) \in L_1$

$u(s, 1) \in L_2$

$u$ is $J$-holomorphic ($\mathbb{R} \times [0, 1] \subset \mathbb{C}$).

$\lim_{s \to -\infty} u(s, t) = y$

$\lim_{s \to +\infty} u(s, t) = x$

$\mathbb{R}$-action is translation in $s$-direction.

Thm: For generic $J$, $\mathcal{M}(y, x)$ is a compact $0$-dim'l mfd.

(so $\partial$ is well-defined). N.B. We require $|y| = k\cdot|y| + 1$ for this.

And $\delta^2 = 0$. 
Composition of morphisms:

Product: \( f_1 \circ \text{Hom}(L_1, L_2) \)

\( f_2 \circ \text{Hom}(L_2, L_3) \)

\[
h \circ g = \sum_{h \in \text{Hom}(L_3, L_1)} \# M(a, b, c) \cdot h
\]

where \( M(a, b, c) \) counts

**J-holomorphic maps**

\[ u: D \to M \]

\[ u(p_1) = a \]

\[ u(p_2) = b \]

\[ u(p_3) = c \]

**Fukaya Category**

\( A_\infty \) category: operations \( m_k \), associativity relations.

E.g.

\[ m_3(a_1, a_2, a_3) \in \text{Hom}(a_1, a_2, a_3) \]

E.g. \( X = \text{top space} \)

\[ \text{Ob} X = \text{pts of } X \]

\[ \text{Mor}_X = \text{chains in } \mathbb{R} \cdot (P(X, Y)), \]

\[ m_1 = \text{boundary operator} \]

\[ m_2 = \text{concatenation} \]

\[ m_3 = \text{associativity homotopy} \]
\[ \mathcal{C} = \text{A}_\infty \text{ category} \]

\[ \text{H}(\mathcal{C}) = \text{same objects, } \text{Hom}(a, b) = \text{H}_\infty(\text{Hom}_\omega(a, b), \text{m}_\infty) \]

\[ \text{composition } = \text{m}_\infty. \]

**Fukaya category**

**Objects**: graded exact Lagrangians

\[ \text{Hom}(D_1, D_2) = \mathbb{Q} \cap \phi^*(L_2) \]

\[ \text{m}_1 = \text{Floer differential} \]

\[ \text{m}_\infty = \text{pants product} \]

\[ \text{m}_\infty(a_1, \ldots, a_k) = \sum_{I \in \text{In} D_1} \phi^*(M(\text{am}, \ldots, a_k, \epsilon)/\pi) \]

\[ \epsilon : D \to M \text{ J-hol} \]

Marked points on \( \partial D \)…

**Directed Fukaya Categories**

**Lefschetz filtration**

\[ \begin{array}{c}
\text{Choose a \textit{basis of vanishing paths}.} \\
\text{vanishing chimbles } T_1, T_2, T_3 \\
\text{(ordered)} \quad V = \text{vanishing cycles} \\
\end{array} \]

**Objects**: \( T_i \)

\[ \text{Hom}(T_i, T_j) = \mathbb{Q}^{\text{H}^1, \text{d}H(T_j)} \text{ s.t. } H \text{ is a Hamiltonian s.t. } H = \pi^* \text{ near } \partial D. \]

\[ \Rightarrow \text{Hom}(T_i, T_j) = 0 \quad \text{if } j < i \]

\[ \text{Hom}(T_i, T_j) = \mathbb{Q}^{\text{H}^1 \times \text{d}H(T_j)} \]

\[ \Rightarrow \text{Hom}(T_i, T_i) = \mathbb{Q}. \]

\[ \text{m}_\infty = \text{same as before.} \]
Another definition: \( V_1, \ldots, V_k \subseteq E \).

\[
\text{Hom}(V_i, V_j) = \begin{cases} 
\mathbb{C} & i > j \\
\mathbb{C} & i = j \\
0 & \text{else}
\end{cases}
\]

\( m_k = \text{induced product as above, completely inside the fibre.} \)

These definitions are equivalent because J-holomorphic curves have to remain inside the fibre by the maximum principle.