Def. A m-oidal $\mathbb{A}^m$-category consists of the following:
1. Objects $L_1, L_2, L_3$.
2. $V_{ij}$ set of objects, $F_{ij} = \text{Hom}(L_i, L_j)$.
3. A complete collection of chain complexes $\mathbb{C}$, global sections $\mathbb{V} = \mathbb{R} \Gamma(M^2, \mathcal{E}_n^2)$. See below!

Define a sub $\mathcal{E}_n^2$ on $M^2$

$\overline{\mathcal{E}_n^2}$ as follows:
$$(\overline{\mathcal{E}_n^2})_n = \text{Hom}(F_{i_2} \otimes \ldots \otimes F_{i_{n-1}}, F_{i_n})$$
$$(\overline{\mathcal{E}_n^2})_0 = (\mathbb{V}_{31}^{L_1} \otimes \mathbb{V}_{34}^{L_2} \otimes \mathbb{V}_{35}^{L_3})_{D_{\sigma}} \otimes (\mathbb{V}_{32}^{L_2} \otimes \mathbb{V}_{35}^{L_3})_{D_{\tau}}$$

Now $D'$ on right is near:
$$(\overline{\mathcal{E}_n^2})' = \infty$$

So we have a sharp (really careful) m-oidal structure, need to know how it maps from one stratum to another.

$$(\overline{\mathcal{E}_n^2})'_0 = \text{Hom}(F_{i_2} \otimes \ldots \otimes F_{i_{n-1}}, F_{i_n})$$
$$\text{Hom}(F_{i_2} \otimes F_{i_3} \otimes F_{i_5}, F_{i_1})$$
$$\text{Hom}(F_{i_5} \otimes F_{i_3}, F_{i_1})$$

Notation $n \geq 3$:
Given "labels" $L_1, \ldots, L_3$.
Define a moduli space $M^2 = \overline{\mathcal{E}_n^2}$ stable discs satisfying labeled by $L_1, \ldots, L_3$
e.g. $L_1, L_2$
point to point of $M^2$
Example: $n=4$

$g_{n,4}^{1,3,4}$

cross-ratio

By the way, slogan: every disk has at least 3 special points.

For us, the rough picture is this: we have a symplectic manifold $\mathcal{M}$.

1. The objects are Lagrangian subspaces $L \in \mathcal{M}$.
2. Given $L_i, L_j \in \mathcal{M}$, $F_{ij}$ is a free module generated by intersection points $x \in L_i \cap L_j$ (prefer: there are no degrees or differentials)

$Z = L_1, \ldots, L_n$, $\mathcal{M}^Z$ $\mathbb{R}^3$

Roughly speaking, to give a global section $\mathcal{M}^Z$ of $\mathcal{E}_Z$, need to assign an integer to each picture labelled by Lagrangian intersections.

Idea: These integers measure holomorphic maps $D \rightarrow \mathcal{M}$ compatible with boundary data.

$\mathcal{M}^Z \rightarrow \mathcal{M}_Z$

Facts: $\mathcal{M}^Z$ is a smooth, compact manifold w/ corners, equipped with an orientation, of same dimension as $\mathcal{M}_Z$.

Note: $\mathcal{E}_Z$ smooth space of $D \rightarrow \mathcal{M}$

$\mathcal{M}_Z$ equipped w/ a holomorphic map $D \rightarrow \mathcal{M}$, compatible w/ boundary data $\partial D$. 
M becomes compact if:

1. We impose a bound on the energy of \( D \to M \).

2. We allow "bubbling":

   \[ 0 \to M \]

   \[ \infty \to M \]

   Flavors of bubbling:

   1. Disk can degenerate into a stable disk.

      \[ (\infty) \to (\infty) \quad \text{Good} \checkmark \]

   2. Disk bubbles off a holomorphic sphere.

      \[ (\infty) \to (\infty) \quad \text{not really problematic} \]

   3. Disks bubble off holomorphic strips.

      \[ (\infty) \to (\infty) \]

     (These are important, give rise to differentials in Floer complex).

   4. Disks bubble off disks w/ one distinguished point

      \[ (\infty) \to (\infty) \quad \text{result is "curvature"} \]

In many cases, you can choose \((M, \omega)\) & Lagrangian \(L\):

rule out curvature.  get \( [\omega] \in \text{H}^2(M, \mathbb{R}) \)

if nontrivial \( \Omega^2, \omega \cdot \text{H}^2(M, \mathbb{R}) \) determines \( \text{holomorphic if } [\omega] = 0 \).
Ex: $n = \text{surface of genus } g > 0$

$\pi_1 L \leftrightarrow \pi_1 M$

Def: A manifold is pseudo-stable if every component has at least 2 special points.

$M_{\tilde{g}}$ = like $M_{\bar{g}}$ but allow pseudo-stable maps

$M_{\tilde{g}}^{ps}$ = like $M_{\tilde{g}}$ but allow $p.s.$ (The map is stable, e.g., any two-point component has nonzero energy).

Goal: Understand "orientation" and dimension of $M_{\tilde{g}}^{ps}$

X a manifold: Preserve duality.

In last case:

$H^*_c(X) \cong H_{n-\ast}(X)$

If X is non-compact,

$H^*_c(X) \cong H_{n-\ast}(X)$

If X has boundary/corners:

$H^*_c(X) \cong H_{n-\ast}(X, \partial X)$

If X is not oriented,

$H^*_c(X) \cong H_{n-\ast}(X, \partial X; k_X) \otimes k_X^X$.

R comm. ring,
Pic $R$ = category of trivial graded line over $R$

(automorphisms are isos.)

Abuse of notation: Pic $R$ = classifying space of Pic $R$

$\pi_0 \text{Pic } R = \mathbb{Z}$ (what degree vector is in)

$\pi_1 \text{Pic } R = R^\ast$ (can be a ring spectrum too, in which case it has higher $\pi_n$).
$X \rightarrow \text{Pic } \mathbb{R}$

A sheaf on $X$ whose stalks are formal graded lines over $\mathbb{R}$.

Example: Orientation sheaf of $X$.

$X \rightarrow \text{Pic } \mathbb{R}$

Lagrange Grassmannian

$\tilde{\mathcal{M}} = \bigoplus \mathcal{M}_L \rightarrow \mathcal{B} \mathcal{O}(n) \rightarrow \mathcal{B}O \rightarrow \ast$

Complexification

$\mathcal{B}U(n)$

$\mathcal{B}U(\mathbb{R})$

$\mathcal{B}^2(\mathbb{Z} \times \mathbb{R})$

Summary 1:

There is a canonical 2-gerbe $\tilde{\alpha}$ over $\mathcal{M}$.

Now suppose:

$\tilde{\mathcal{M}} \rightarrow \ast$  \(\tilde{\alpha}\) has a canonical trivialization over every point.

$L \rightarrow \mathcal{M} \rightarrow \ast$  \(\tilde{\alpha}\) has a canonical trivialization on every fiber.
To define Fukaya category (over $R$) we need the following:

1. Choose a trivialization of $\xi$ over $M$. This gives for every Lagrangian $L \subseteq M$, a 1-gerbe $\beta_L$ on $L$ (by composing this with canonical trivialization on $L$).

2. Objects of Fukaya category

$$\text{Obj: } R = \mathbb{Z}/2.$$ 

$$\text{Pic}_R \cong \mathbb{Z} \cdot \mathbb{B}^2 \text{Pic}_R \cong K(\mathbb{Z},2) \cong \mathbb{C}P^\infty.$$ 

$$M \to BU(n) \to K(\mathbb{Z},2)$$

classified by $2\zeta(M) = H^2(M; \mathbb{Z})$.

**Ex: $R = \mathbb{Z}$**

$$\mathbb{K}(\mathbb{Z}_{\geq 1}) \to \text{Pic}_R \to \mathbb{Z} \to K(\mathbb{Z}/3).$$

$$M \to \mathbb{B}^2 \text{Pic}_R$$

$$2\zeta(M) \to K(\mathbb{Z},2).$$

**Ex: $R = \mathbb{Z}/2$**

$$\beta_L : L \to \mathbb{B} \text{Pic}_L \cong K(\mathbb{Z},1) \cong S^1$$

$$L \to L \to S^1.$$

Note that

$$M \to$$

$$L \subseteq M \to BU(n) \to BU \to BU/\text{det} \rightarrow 12$$

$$\to BU(1) \to \mathbb{B}^2(\mathbb{Z} \times \mathbb{Z}) \to \mathbb{B}^2 \text{Pic}_R.$$
Suppose $M$ is simply connected, choose a base point $x$.

What you get is a 2-fold map

$$\mathbb{Z}^2 M \to \mathbb{Z} \times \\text{BO}$$

To this data, you can associate a Thom spectrum,

$$T(\mathbb{Z}^2 M) \in \text{ny spectrum}$$

universal $R$ for which $x$ has a trivialization.

Clockwise rotation, say, the two trivializations of $x$ agree on $L_2 \cap L_1$.

If I have trivializations of $\beta_1$ and $\beta_2$, get an element

$$e_{x_1, x_2} \in \text{Pic } R$$

(Morriwides?

More careful defn of Fukaya.

Objects: Lagrangians L, trivializations of $\beta_L$.

$L_i, L_j \to \otimes e^{L_i, L_j}$

Get a sheaf $\mathcal{E}^2$ over $\mathbb{Z}^{n+1}$.

$\mathcal{M}_2 \to \mathcal{M}_2^{\mathbb{Z}/p}$

Goal: Understand "orientations" of $\mathbb{Z}/p$. \[ \text{ps.} \]
$K_{m_{\text{ps}}} \cong f^* K_{m_{\text{ps}}} \circ K_{m_{\text{ps}}} \circ K_{m_{\text{ps}}}$

$x_{0.1} \quad P \quad M$

Claim: There is a canonical isomorphism:

$$(K_{m_{\text{ps}}})^* \cong (e_{x_{1/2}} \circ \ldots \circ e_{x_{m-n}} \circ (e_{x_{n-1}})^*)$$

What we get:

$K_{m_{\text{ps}}} \rightarrow f^* \mathcal{E}_L$

There exists a fundamental class $[\tilde{m}] \in H_0(\tilde{m}_{\text{ps}}; f^* K_{m_{\text{ps}}} \circ K_{m_{\text{ps}}})$

$H_0(\tilde{m}_{\text{ps}}; f^* K_{m_{\text{ps}}} \circ K_{m_{\text{ps}}})$

$H^0(M_{\text{ps}}, \mathcal{E}_L)$

$H^0(M_{\text{ps}}, \mathcal{E}_L)$

$M_{\text{ps}} \rightarrow M^2$

If you try to push forward a section to $M^2$, run into Floer differential stuff.