Self-dual obstruction theory (Behrend)

Model: \( f: X \to E \)

\[ M = (df)^{-1}(0) \]

\[ M \] has \( \operatorname{verdim} \theta \)!

Suppose \( \dim M > 0 \), \( M \) smooth.

\[
0 \to TM \to T^*A|_M \xrightarrow{D(df)} T^*_AM \to \text{color} \to 0
\]

Hessian (\( \tilde{D}_{ij}^f \)), symmetric

Dual map \( T^*_AM \xleftarrow{\text{color}} TA|_M \) is also \( D(df) \)

\[ \text{color} = T^*_AM. \]

Deformations/obstructions are dual (def'n of self-dual obs.

Perturb \( f \) by a function \( g \) on \( M \), \( df + \varepsilon g \) \( \to 0 \) theory)

\[ = \text{zeros of } dg \text{ in } M. \]

Virtual cycle should be \( \cdim M \left( T^*M \right) = (-1)^{\dim M} e(M) \)

In general (\( M \) self-dual obs. theory, not necessarily smooth),

Behrend shows that \( [M]^{\text{virt}} \in \text{Ho}(M) \) is

\[ e(M, X^E) = \sum_{n \in \mathbb{Z}} e((X^E)^{-1}(n)) \text{ for some constructible \( f: M \to X^E \).} \]
$X^6 = (-1)^\dim M$ where $M$ is smooth.

Depends only on the local analytic str. of $M$!

For simplicity, ignore $X^6$, signs, just work w/ e. (very queer, can't compute w/o $F_\nu$.

E.g. MNOP — do def. theory not of subscheme $Z \subset X$, not of ideal sheaf $I_Z$, but of sheaf $\mathcal{O}_Z$.

$$T_{\mathcal{O}_Z}(X, \beta) = \text{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \xrightarrow{\text{some dual}}$$

$$\text{Ob}_Z = \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \xrightarrow{\text{CCY}^3}$$

Stable pairs: def. theory not as a pair, but as the trivial determinant $Z$: $m_Z \in D^b(X)$.

$$I = \left\{ \mathcal{O}_X \xrightarrow{S} F \right\} \in D^b(X)$$

$$T_{\mathcal{O}_X}(X, \beta) = \text{Ext}^1(I, I) \xrightarrow{\text{dual}}$$

$$\text{Ob}_I = \text{Ext}^2(I, I) \xrightarrow{\text{dual}}$$

Work with a fixed CM curve $C \subset X$, $\pi(C) = 1-9$

$$\text{Im}(X, C) = \text{Im} + \nu (X, \beta)$$

 locus of subschemes whose largest CM subscheme is $C$.

$Z = C \cup$ embedded pts. $\cup$ free pts. $\cup$ $C$.
Similarly, $P_n(X, c) = P_{n+1-g}(X, \beta)$

pairs supp. on $C$

\[
\frac{Z_{mn\rho, \beta}}{Z_{mn, \rho, \beta}} = Z_{\rho, \beta}
\]

follow from \( \text{constancy } \Gamma_{\rho, \beta} \)

\[\Gamma_{\rho, \beta} : P_{\rho, \beta} \equiv P_{\rho, \beta} + P_{\rho-\beta} \cdot e(\text{Hilb}^\rho X) + \ldots + e(\text{Hilb}^\rho X) + \ldots
\]

c. etc.

\text{Wall crossing: } Z = C \cup \{P_i\}

\[\begin{align*}
\text{Except } & P_i \\
& P_{\rho - 1} \rightarrow g^* \rightarrow g_c \\
& O_{P_i}(-1) \rightarrow g \rightarrow g_c \text{.}
\end{align*}
\]

Move across a wall in space of stab conditions, so that

"slope" of $O_{P_i}[-1]$ crosses that of $g_c$.

When bigger, \( \odot \) destabilizes $g_2$ & is stable object?

However, extensions $g_c \rightarrow ?? \rightarrow O_{P_i}$

are now stable.

(\text{Can work at $b_1$ LES that } Ext^1(O_{P_i}, [-1], g_c) =

\text{Exts } ?? \text{ are of the form:}

\[\begin{align*}
I^* &= \{O \rightarrow F \} \\
O_{P_i} \rightarrow \text{nontrivial } & \text{c.}
\end{align*}\]
Stable objects change as we cross wall from $\text{In}(x, \beta)$ to $\text{P}_{\mu}(x, \beta)$. Invariants change?

Model: Sheaves $E$ which can become unstable or cross a wall in Kähler cone

$0 \to A \to E \to B \to 0$

slope $A$ crosses slope $B$. but $A, B$ do not decompose, remain stable on both sides of wall.

$\Rightarrow \text{Hom}(A, B) = 0 = \text{Ext}_{\mu}^0(B, A)$. So by Serre Duality, $\text{Ext}_{\mu}^3(A, B) = 0$ etc.

so only have:

$\text{Ext}^1(A, B): 0 \to B \to F \to A \to 0$

$\text{Ext}^1(B, A) = \text{Ext}^2(A, B): 0 \to A \to E \to B \to 0$

when we cross a wall, lose all extensions $\Rightarrow (\text{P}^{\text{Ext}^1(A, B)})$

gain all extensions $\Rightarrow (\text{P}^{\text{Ext}^1(B, A)})$

R.R.

Difference $\Rightarrow \chi(B, A)$

$\chi = \chi(B, A)$

Difference in invariants as cross wall is $\chi(B, A) e(\text{det}(B))$.

Our case: $A = \mathbb{C}^n[-1]$ split up, semistable vector,

$B = \mathbb{C}^n$ stable.
Our case: branch of stacks.

A free pt.

\[ I_{t,c} = P_{t,c} + e(X) \]

Let's see this using geometry in case C smooth.

\[ p \in X \setminus C \mapsto 1 \to I_{t,c} \]

\[ p \in C \mapsto 0 \to P_{t,c} \]

\[ 2 = e(P^1) \to I_{t,c} \]

\[ 1 \to P_{t,c} \]

See that \[ I_{t,c} = e(X \setminus C) + 2 e(C) \]

\[ P_{t,c} = e(C) \]

\[ e(P^1) \]

\[ e(X) \]

\[ I_{t,c} = e(X) + P_{t,c} \]

\[ I_{t,c} - P_{t,c} = e(X) \]

General \( c, p \in X \)

\[ e(P (\text{Hom}(I_c, O_p))) \to I_{t,1} \]

\[ \text{Hom}(I_c, O_p) \]

\[ e(P (\text{Ext}^1(I_c, O_p))) \to P_{t,c} \]

\[ \text{Ext}^1(I_c, O_p) \]

By Riem. Roch, \[ \text{hom}(I_c, O_p) - \text{ext}^1(I_c, O_p) = 1 \].
\[ D_{\eta} \text{ in } \mathfrak{m} \text{ is } +1 \text{ for } X. \]

\[ X \in A \times \mathfrak{m}. \text{ in } \mathfrak{m} \text{ is } e(X). \]