Miami '09 — Thomas I:

Curve Counting and Derived Categories

Stable Pairs

1. $GW, MNOP, \text{ virtual cycles, stable pairs}$
2. wall crossing
3. $G \times \mathbb{BPS}$

$x$ smooth proj. $(\text{3-fold, i.e. } K_x = 0_x$) (can extend to all 3-folds, ...)

All "things" live in families of virtual dim $0$,

- slags, vials,
- curves, sheaves,
- surfaces,
- critical pts. of a function

Hope to define invariants by "counting" them.

- compactness of moduli space < finite invar < deformation invar
  (spts pts. maps)

- transversality/virtual cycles

These lectures — count hol. curves in $X$

$GW$ theory counts stable maps:

\[ \text{keep curve nice, nodal curve } \to X \]

"Stable" — finite autos $\Rightarrow$ invariants $\in \mathbb{Q}$. In $GW$ theory limit is just

\[ \infty \to \text{ branch normalization map } \to \text{ embedded at crossing pt.} \]
2:1 double cover
\( \mathbb{Z}/2 \)-action.
Counts as \( \frac{1}{2} \) to \( GW \)-
\( \text{limit} \).

(student): How does this count as \( \frac{1}{2} \)? Was before as \( 1 \)?
Look at families -- missed \( p^2 \).

(2) Subschemes

MNOP theory
Hilbert scheme \((X)\)
Map embedding
(curve bud.

\[ \begin{array}{c}
\text{local model} \\
\text{x=0} = 2 \\
y=0, z=t \rightarrow 0.
\end{array} \]

\[ (x, z) \cdot (y, z-t) = (xy, yz, x(z-t), z(z-t)) \]

\[ t \rightarrow 0 \]

\( (xy, z) \Rightarrow (xy, yz, xz, z^2) \)
\( z \) ideal, but \( z \) any elt. of maximal ideal

\( z \) ideal.
Genus change,
free pts. $\mathcal{C} \xrightarrow{\chi} \mathcal{X} \xrightarrow{\alpha} \mathcal{C}^*$

\[ g^+ = 1 \]

\[ \# \text{free pts.} + = 1 \]

Subschemes contain zero-jet subscheme.
No autos $\Rightarrow \text{Injts. } \in \mathbb{Z}$
(based on #free pts. by Chern class)

\[ \text{thickened line} \]
\[ x^2 = 0 = z \]

**MNP Conjecture:**

\[ Z_{GW}(u) = \sum_{g \geq 0} N^g \cdot u^{2g-2} \]

$\beta \in \mathbb{H}_2(X)$

other side:

\[ n = \chi(O_2) = 1 - g(c) + \# \text{(free pts.)} \]

GW theory

subscheme $Z = C \cup \text{free pts.}$

\[ Z_{\text{MNP}, \beta}(t) = \sum_{n} I_{n, \beta} t^n \]

\[ Z_{\text{MNP}, 0}(t) = \sum_{n} I_{n, 0} t^n \]

\[ \frac{Z_{\text{MNP}, \beta}(t)}{Z_{\text{MNP}, 0}(t)} = \frac{Z_{GW, \beta}(u)}{-e^{iu} - t} \]

Some sort of analysis coming along.
\( t(1+t)^{-2} = t - 2t^2 + 3t^3 - 4t^4 + \cdots \)

\( \text{inv. under } t \mapsto \gamma t \)

Lawmot expansion, NOT inv.

Stable pairs

\( (S, S) \in H^0(F) \)

satisfying:

\( \exists \cdot F \text{ is pure} \)

\( \exists \cdot S \text{ has 0-dim'l coker.} \)

\( [F] = \beta. \)

Examples:

- \( (\mathcal{O}_C, 1) \)
- \( (\mathcal{O}_C(D), S_D) \)
  - curve + free pts. on \( C \).
- \( (\mathcal{O}_C, 1) \) limit of picture (8)

\( \mathcal{O}_{C_1} \cup \mathcal{O}_{C_2} \rightleftharpoons \mathcal{O}_{c_1} \cup \mathcal{O}_{c_2} \)

\( (\mathcal{O}_{C_1} \cup \mathcal{O}_{C_2}, (1,1)) \)

\( \text{coker } \mathcal{O}_p \)

\( (\mathcal{O}_{c_1}(p), S_p) \)

\( \mathcal{O}_{C_1} \cup \mathcal{O}_{C_2} \rightleftharpoons \mathcal{O}_{c_1} \cup \mathcal{O}_{c_2} \)

\( (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}, (1,1)) \)

\( \text{coker } \mathcal{O}_p \)

\( (\mathcal{O}_{c_1}(p), S_p) \)
pure $\Rightarrow C$ is Cohen-Macaulay, i.e.,
- no embedded pts.
- no free pts.

(when $C$ is Gorenstein, stable pair $\iff (x, D)$ pt. of $\text{Hilb} C$)

why Gorenstein? need $w_C$.

Aside: $\text{Ext}^i_C(\mathcal{O}, \mathcal{O}_C) \cong H^0(Q \otimes \mathcal{O}_C)^i$

$\text{valuer}(\text{pair})$

$n = \chi(F) = \chi - \text{g}(C) + \#(\text{free pts.})$

$Z_{p, \beta} = \sum_{n} P_{n, \beta} t^n$

Conjecture: $Z_{p, \beta}(t) = \frac{Z_{\text{manop}, \beta}(t)}{Z_{\text{manop}, 0}(t)} = Z_{\text{GW}, \beta}(u)$

and again, this $J$ is the Laurent expansion of a ratio function that's invariant under $t \leftrightarrow 1/t$.

Note: Over a good component, ideal of sheaf

$$(\mathcal{O}_C, 1) \xrightarrow{\mathcal{J}} \mathcal{O}_x \xrightarrow{\mathcal{O}_C} \mathcal{F}_x$$

for good $C$ equiv. data $\mathcal{J}$