joint w/ Donagi and Simpson

- Relationship between non-abel. hodge theory (nht)
  & geometric Langlands correspondence (glc).
- An example. (all \( \mathbb{C} \! \setminus \! \ast \))

1. GLC for \( \text{GL}_n \) sticky (D-M-stack).
   
   - Everything / \( \mathbb{C} \)
   - \( C \)-smooth, proper (DM) curve

\( \text{Bun} \) = moduli stack of rk. \( n \) v.s. on \( C \).

\( \text{Bun} = \text{Bun} / \mathbb{C}^* \) = moduli stack of
   \( (\text{Hilb generic automorphism}) \) bundles

\( \text{GLC} \): Suppose \( V = (V, \nabla) \) algebraic de. n local
   system (irreducible), on \( C \).

Then \( \exists ! \) irreducible D-module \( \mathcal{C}_V \) on \( \text{Bun} \),
   s.t. \( H^k (\mathcal{C}_V) = C_V \otimes \bigotimes_{k=1}^{\infty} V \)

   (Hecke eigensheaf condition).

\( H^k \) - Hecke operator : Integral transform.

\( pk \) \( \text{Hecke} \) \( q^k \)

\( \text{Bun} \) \( \text{Bun} \times \mathbb{C} \)
\[ \text{Hecke}^k = \{ \sigma (V, V') (\beta) \} \]

\[ \begin{array}{c}
\beta : V \rightarrow V' \text{ map of loc. free sheaves,} \\
\text{supp} (\text{coker} \beta) = x \in C. \\
\text{length} (\text{coker} \beta) \\
1 + k = q^k \cdot p^k \\
\end{array} \]

Note: \[ p^k, q^k \text{ - smooth map with fibers } Gr(k, n). \]

One expects that \[ C_V = -\text{quant} B_x. \]

where \( \text{quant} c \) is some quantization for coherent sheaves on \( T^* C. \)

\[ \text{quant} B_x = T^* B_x. \]

1. Non-abelian Hodge theory.

2. Non-abelian Hodge theory, for projective curves.

Then (Carlath-Simpson): \( (X, O_X(1)) \)-projecte.

Then, there is an equivalence \( \vdash\text{dg cats.} \)

\( \text{(cat. of full rank) \local systems on } X \xrightarrow{nabla} \text{(Higgs bundle on } X \text{ w/ } \text{ch}_1 = \text{ch}_2 = 0.} \)
Higgs sheaf on $\mathcal{X} : (F, \phi)

\begin{align*}
F \in \text{coh}(\mathcal{X}) \\
\phi : F & \to F \otimes L_x \\
\psi \circ \phi & = 0 \\
\psi : T_x \otimes F & \to F \otimes L_x \\
\psi : T_x \otimes F & \to F \\
\psi : T_x \otimes F & \to F
\end{align*}

Fact: (Hitchin, Faltens, ...) $Higgs_c = T^* Bun$

If $V = \text{rk} u$ local system

$\rightarrow$ $\text{h}^1_c(V) = \text{(rank } u \text{- Higgs bundle)}$

$Higgs_0 = T^* Bun$

$\exists \text{ universal map }$

$T^* Bun \searrow T^* Bun$
\[ h: T^v B_n \rightarrow B \quad \beta = \text{H}^0(K_c) \otimes \]
\[ h: T^v B_n \rightarrow B \quad \otimes \text{H}^0(K_c) \otimes. \]
\[ h(E, \theta) = (h \theta, \text{tr}(\theta^n)) \]

If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in B, \)
\[ \bar{\alpha} : x_n + \alpha_1 x_n^{-1} \quad -\alpha_n = \text{O}(\mathcal{L}). \]

Fibers \( h^{-1}(\alpha) = \text{Pic}(\bar{\alpha}) \times \text{Pic} \text{stack} \)
\[ h^{-1}(\alpha) = \text{Pic}(\bar{\alpha}) = \text{Pic}(\bar{\alpha}) \]

If \( \bar{\alpha} \) is smooth \( \Rightarrow \)
we have a pushout sheaf \( P \rightarrow \text{Pic}(\bar{\alpha}) \times \text{Pic}(\bar{\alpha}). \)
\[ FM: D^b(\text{Pic}) \rightarrow D^b(\text{Pic}) \]
\[ D^b(\text{Pic}) \xrightarrow{\text{h}} D^b(\text{Pic}). \]

Thus, if \( B^* \subset B \Rightarrow \) have
\[ h: (T^v B_n)^* \xrightarrow{\text{geom}} (T^v B_n)^* \]
and \( \text{fh} = \text{fh}_{\gamma}: D^b((T^v B_n)^*) \rightarrow (T^v B_n)^* \)
\[ (T^v B_n)^* \xrightarrow{\text{h}} (T^v B_n)^*. \]
$\text{Fm. quant}_{\mathbb{C}}(V) = \text{Fm}(\Omega_{\text{nah}_c}(V))$

\[ \Rightarrow \text{Dr}_c(T^*\mathcal{R}_c)! \]

In fact this is a sheaf

$\text{Fm. quant}_{\mathbb{C}}^{-1}(V) = i_*L_V$

\[ i_* : \text{Pic}(\mathcal{E}_\alpha) \rightarrow T^*\mathcal{B}_\mathbb{R} \]

$L_V \in \text{Pic}^0(\text{Pic}(\mathcal{E}_\alpha))$ corresponding to $\text{m}_c(V) \in \text{Pic}(\mathcal{E}_\alpha)$

Note: $i_*L_V$ is a Hecke eigensheaf for a classical limit version of the Hecke.

Note: $i_*L_V$ is finite over $\mathcal{B}_\mathbb{R}$

$\Rightarrow$ Higgs sheaf on $\mathcal{B}_\mathbb{R}$

Idea: quant $\mathcal{B}_\mathbb{R} = \text{nah} \mathcal{B}_\mathbb{R}$

Two problems: $\mathcal{B}_\mathbb{R}$ not a projective variety.

1. $\mathcal{B}_\mathbb{R}$ is an Artin stack which is not of finite type.

2. Not proper

Way out: reduce the question to a question about (log) varieties.
Thm (Machita): If \( V \) is quasi-projective,

\[
\begin{align*}
Y & \to X \quad \text{compactifications} \\
p & \to U, \quad p|_U = 1
\end{align*}
\]

and \( D = X \setminus U (X, D) \) - as before,

then there is equivalence of \( \text{semistable D-modules over nilpotent \( D \)-endoms, semistable parabolic data) \to (\) \}

\( \text{local sys on } (X, D) \)