Miami '09: Pandharipande

Curve counting: \( g = 0 \) on surface

KS

\[ \log CY. \]

\[ \text{Curvature } C \rightarrow \mathbb{P}^2 \setminus C \]

\[ \text{alg. map: } C \rightarrow \mathbb{P}^1 \]

\[ \text{Count dimensions} \]

\[ \text{generic deg } 3 \text{ map nowhere at 39 points, want them all piled up!} \]

\[ 3 \text{-1} = \text{dim } \mathcal{M}_0(\mathbb{P}^2, 3) \]

\[ \Rightarrow 0 \text{ dim 'I problem } \rightarrow \text{count} \# \]

N. Takahashi 1999 relates this to \( g = 0 \) moment of local \( \mathbb{CP}^2 \).
Ex: \[ \begin{array}{c}
\text{This sort of cubic very trix - } \\
P^2 \text{ divisor with } \\
\text{torsion.}
\end{array} \]

Can't quite do this (can't have an 3d tangency to a line.)

\[ (S, D) \text{ } \beta \in H_2(S, \mathbb{Z}) \]

\[ \beta \circ \zeta(S) = (\beta \circ D, \text{ so } (S, D)) \text{ is } \]

\[ \log CY \text{ w.r.t. } \beta. \]

\[ \nabla \text{ count great D curves w/ full } \]

\[ \text{contact at a single } p \text{ of } D. \]

\[ C^\times \times C^\times \text{ at bottom, } \]

\[ \text{& group } \]

\[ \text{Aut } Gr = GL(2, \mathbb{Z}) \]

\[ \text{Aut } (C^\times \times C^\times) \text{ (alg. w/ b) } \text{ not many more } \]

\[ \text{alg. translations} \]

Formal 1-param families of Aut.

\[ C^\times \times C^\times = \text{Spec } (C[z, x^{-1}, y, y^{-1}]) \]

\[ A = \text{Aut}_C \Gamma (C[x, x^{-1}, y, y^{-1}][t]) \text{ formal 1-param } \]

\[ \text{Aut}_3. \]
\( (a,b) \in \mathbb{Z}^2 \) \( f = 1 + t \cdot x^a y^b \cdot g \)

where \( g \in C[x^a y^b \mathbb{J}^k] \)

Given this, \( g \neq 1 \).

\( \Theta_{(a,b)} f : (x) = x^f - b \cdot \Theta_{(a,b)} f = (x^f - b)^{-1} \)

\( \Theta_{(a,b)} f = x^a y^b \)

A.

\( \text{N.B.}: (x^a y^b) = x^a y^b \)

\( \text{TVG} \subset \text{A} \) (top. vect. gp.)

gen. by all such \( \Theta_{(a,b)}, f \)

Note that on \( \mathbb{C}^x \times \mathbb{C}^y \), \( \omega = \frac{d x}{x} \wedge \frac{d y}{y} \), that

\( \Theta_{(a,b)} f (\omega) = \omega \Rightarrow \frac{d x}{x} \wedge \frac{d y}{y} \subset \text{A} \)

\( \text{TVG} \subset \text{Aut} \text{D} \quad \subset \text{A} \)

\[ \begin{array}{c|c|c}
(0,1) & (1+3x)^{k_2} & (0,1) \\
\hline
(1,0) & \Theta_{(1,0)} (1+3x)^{k_2} & (0,1) \\
\hline
(0,1) & & \\
\end{array} \]

\( \text{want to eventually discuss contrib.}

\( \text{of these 2 cols.} \)
Lemma: (K-S) Given a path ordered product as above, one can always add more arrows $s_i$, resulting in a path ordered product that is the identity, primitive.

In our case:

$T_{2, k} = \Theta(a_{i, 1}, \ldots, a_{i, k})$

$\Theta(a_{i, 1}, \ldots, a_{i, k}) f(a_{i, k})$

What is $(a, b) \cdot f(a, b)$?

$(0, 1)$, $x$, $(a, b)$ primitive.

3 dark vectors are the fur of a tiger surface, weighted $\mathbb{P}^2$. 

(first part)
\( \tau_{ab} \) is a genus series for these genus 0 log CY cuts associated to exactly this geometry. \( Q \) is ordered partition \( Q = (q_1, q_2, \ldots, q_2) \)

\[ P = (p_1, -p_2) \quad \sum p_i = ak, \quad q_i > 0 \]

\[ P' = (p_1', -p_2') \quad \sum p_i' = bk. \]

Want to define a class \( \beta_k \in H_2(P, q, b) \)

\[ \beta_k \cdot D_{at} = k. \quad \text{want to only meet } D_{at}. \]

\[ \beta_k \cdot D_2 = bk \quad \text{once, but then might meet other any times.} \]

\[ \beta_k \cdot D_1 = ak \]

\( \tilde{X} \) blowup \( D_2 \)

\[ \tilde{\beta}_k = \pi^* \beta_k - \sum p_i E_i \]

Now, \( \tilde{\beta}_k \) is the usual log CY push
So by looking at \((X, \tilde{D}_{x+y}), \tilde{B}_n, \gamma + \text{a number } N(p, \ell, p_2)\) \(\text{partition}\).

Now, \(\log f_{a,b} = \sum_{k \geq 1} k C_{a,b}^k \left( \frac{h_1}{l_1} \frac{h_2}{l_2} \right) (1+x)^{ak} (1+y)^{bk} \)

\(C_{a,b}^k \left( \frac{h_1}{l_1} \frac{h_2}{l_2} \right) = \sum_{\text{Partitions}} N(a,b) \left( P, P' \right) \)

\(|P_1| = ak \), \(|P_2| = bk \)

Rule: before working over \(\mathbb{C}[t, \bar{t}]\),

For an exact formula, should be working over

\(\mathbb{C}[t_1, \ldots, t_{l_1}, s_1, \ldots, s_{l_2}]\)

then \((1+t+x)^k\), becomes \(\prod \left( 1+t_i; x \right)\)

\(f = \prod \left( 1+t_i; x \right)\)

Since it's now: \(f = 1+q_1 x+q_2 x^2+\ldots\) \(q_i \in \mathbb{Z}\)

Factor: \(f = (1+x)^{q_1} (1+x^2)^{q_2} (1+x^3)^{q_3} -\ldots\)

\((1+x^3)\) corresponds to a \(p_i, s_i\), along some. guess
subspace of order \(r\), blow up at it, get \(A_{g-1}\) similarly.