Welschinger Invariants

$\mathbb{P}^2$ - 1 pts in general position in $\mathbb{P}^2$

- real rational curve of degree 3 which pass through
  the points of $s$.

$N_d$

- $d=1$: 2 pts, the answer is 1
- $d=2$: 5 points 1, $N_3^C \geq 3$ (lower bound)
- $d=3$: 8 points 8, 10, or 12
  (should be even, $N_3^C$ is even, 6 more/4 less come in pairs)
- $d=4$: 19 points

$N_4^C = 620$

$240 \leq \ldots \leq 620$

- sharp, given by Welschinger
  - not known if sharp in deg. 5!

Obs: if we count appropriately (e.g. use signs), then
- number is independent of $d$ (unlike above).

Welschinger signs:

- $x^2 - y^2 = 0$
- $x^2 - y^2 = 0$

Types of nodes:

- (double pair)
nodes can be imaginary too — ignore these.

- \( x^2 + y^2 \) type, call solitary.

- Sign is \((-1)^s\), where \( s \) is the number of solitary double points.

Denote by \( R^+_d(w) \) the number of curves

\[
R^+_d(w) = R^+_d(w) + R^-_d(w)
\]

even # of solitary pts.

\[\text{odd # of solitary pts.}\]

**Theorem:** (J. Y. Welschinger) : The number

\[
W_d(w) = R^+_d(w) - R^-_d(w)
\]

does not depend on the choice of a (generic) configuration \( w \).

Call it \( W_d \) : Welschinger invariant.

**Remark:**

\[
R_d(w) \geq |W_d|.
\]

In fact, \( W_3 = 8, W_4 = 240 \),

compare w/ \( N_1 \) on prev. page.

This is a specific case of a more general invariant.

There are Welschinger invariants \( W_d \) for some # of imaginary pairs.

Not true that all are positive, though all of conjugates \( W_d \) are > 0.
Thus (Kharchev, Shustin, Ikenberry):

1. \( W_d > 0 \) for any pos. \( d \).

2. \( \lim_{d \to \infty} \frac{\log W_d}{\log N_d} = 1 \).

(Rmk: \( \log N_d = 3d \log d + O(d) \)).

Thus (Kharchev, Shustin, I.)

Let \( \Sigma \) be a toric del Pezzo surface, equipped with a real structure (with non-empty real part).

Let \( D \) be a real ample divisor of \( \Sigma \).

Then, \( \xi (\Sigma) \cdot D = 1 \) real points.

1. \( W_D > 0 \)

2. \( \lim_{d \to \infty} \frac{\log W_d}{\log N_d} = 1 \)

\( \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^k, k = 1, 2, 3 \) blown up at \( 2 \) conjugate imaginary pts.

There are 5 toric real structures:

\( \Sigma_1, \Sigma_{(1,0)}, \Sigma_{(2,0)}, \Sigma_{(0,2)}, \Sigma_{(1,1)} \)

\( \mathbb{P}^1 \times \mathbb{P}^1 \) via \( (z, w) \mapsto (\bar{w}, \bar{z}) \)

(1) Case of \( \mathbb{P}^2 \).

Tropical approach: \( N_d \) (ch.) can be calculated tropically.
Mikhalkin's correspondence theorem

count (up to multi.) until tropical curves, deg. d,
which pass through 3d-1 pts. in general pos. in \( \mathbb{R}^2 \).

\( N_d \) lattice area of dual \( \Delta \) multiply by

try to understand what corresponds to real pts./curves in tropical picture.

\( W_d \) Welschinger multiplicities

\( T \) tropical curve multiply

\( W(T) = 0 \) if \( T \) has at least one even weight.

Otherwise,

\( W(T) = (-1)^i \), where \( i \) = the total number of integer points (lying strictly) in the interior of the triangles of the dual subdivision.

Remark (G. Mikhalkin): For any \( d \),

\[ W_d \equiv N_d \mod 4 \] (clearly from before, they're \( \equiv \mod 2 \).)

- compare calculation, use Pick's formula!
This works verbatim for all the tropical real structures, just change the polygon!

(2) Case of $S$.

(Real: In case of ellipsoid), such a congruence mod 4 is false!

\[ \begin{array}{c|c}
\hline
\text{d} & (d,d) \\
\hline
\end{array} \]

\[ d=2: \]

\[ N_{z,2} = 12, \quad W_{z,2} = 6. \]

Perhaps has to do with this is not a maximal real surface (\( \sum_{i=1}^{2} \alpha_i \sigma_i = \sum_{i=1}^{2} b_i \sigma_i \)).

Can't use the same approach, since there are incompatible real structures acting on picture now!

But not a generic configuration!

But can count with multiplicities to get the right answer.
Example of a trop. curve to count:


drawing of trop. curve

3 "real" corresp. theorem (E. Shustin)
It gives a trop. calculation of Welschinger invariants for $S$.
also gives sum w/ pos. coeffi. - established positively.
Other cases:
   Exactly same approach works for $S(1,0), S(2,0), S(3,2)$
(3) Case of $\mathbb{P}^1 \times \mathbb{P}^1(0,2)$:
Now the symmetry is a central symmetry. All real pts.
concentrated in one pt. too degenerate.
Count real rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$ which have
two imaginary conjugated ordinary multiple points.
If you choose the marked pts. in Caporaso-Harris position
(i.e. all in first $z$-strip)
\[
\begin{array}{cccc}
  * & * & * & * \\
  * & * & * & * \\
  * & * & * & * \\
  * & * & * & * \\
\end{array}
\]
stretch to left, distance in
left bigger $\Rightarrow$ distance
on right.
pts. are in special position, but can cut into blocks, get recursive
formula which agrees w/ formula of Vakil (Caporaso-
Harris type), solving the problem. give result in prop.