Recall: $\hat{X} = V(x_0 \cdots x_n - 1)$

$m$-mirror to $\mathbb{P}^n$

$W = \sum x_i$

universal unfolding $n$

$W^0 = \sum_{i=0}^n t_i W_i$

$M = \text{Spec } \mathbb{C}[t_0, \ldots, t_n]$ 

$R$ a local system on $M \times \mathbb{C}^*$, we fibers $H_n(X_i \mathcal{R}e(z W_i) \ll 0)$.

$1) \Delta_0$, $-\Delta_0$ basis of mult-valued sections of $R$.

$2) A$ section of $R = \mathcal{O}_M \otimes \mathbb{C}^*$

given by integration of $e^{2\pi i f} \cdot \Omega (\frac{dx_0 \cdots dx_n}{x_1 \cdots x_n})$

satisfying:

$3) \text{The monodromy of } \mathcal{L} \rightarrow \mathcal{L} \otimes e^{2\pi i z_i}$ in the basis $\Delta_i$ is $e^{2\pi i (\text{mult}) N_i}$, where

$N = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$4) \text{Identify a fiber of } R \text{ with } \mathcal{C}[x]/(x^{m+1}) \text{ with } \alpha^i \text{ dual to } \Delta_i \text{ (so } \ast N \text{ is just mult by } \alpha \text{) So can write the section determined by } e^{2\pi i f} \cdot \Omega \text{.}
\[
\sum_{i=0}^{\infty} a_i \int e^{2\pi i \sum_{i=0}^{\infty} \Delta_i} \\
= g^{(m+1)d} \sum_{i=0}^{\infty} \Psi_i(t, q) \left( \frac{2\pi i}{q} \right)^i \\
\uparrow \\
(\text{takes out multi-valued ness}) \text{, defining } \Psi_i. \\
\text{where } \Psi_i(t, q) = \delta_{i, 0} + \sum_{j=1}^{\infty} \Psi_{i, j}(t) q^j \\
(\text{normalization condition}) \\
\Psi_i(t) = \Psi_{i, 1}(t) \text{ are the flat coordinates.}
\]

\[\overline{M}_{0, m}(\mathbb{P}^n, d) = \text{moduli space of stable maps to } \mathbb{P}^n \text{ of genus } 0 \text{ m-pointed curves, degree } d.\]

The cotangent line to the m marked points gives a line bundle \[Q_i \text{ on } \overline{M}_{0, m}(\mathbb{P}^n, d)\]

Define \[\Psi_i = \zeta_i(L_i) \in H^2(\overline{M}_{0, m}(\mathbb{P}^n, d), \mathbb{Q}).\]

Let \[\beta_1, \ldots, \beta_m \in H^*(\mathbb{P}^n, \mathbb{Q})\]
\[\langle \psi^{k_1} \beta_1, \ldots, \psi^{k_m} \beta_m \rangle^d = \int \wedge^i (\psi_i^{k_i} \wedge \text{ev}_i^* \beta_i)\]

The generalization of the usual GW-invariants. \[\overline{M}_{0, m}(\mathbb{P}^n, d)\]
Put into a generating fn:  
(Gravitational descendants)

Given $T_i$: Let $T_i$ be the primitive pos. generator of $H^2(P^n, \mathbb{Z})$

Then,

$$J_p(y_0, \ldots, y_n, q) = e^{2(y_0 T_0 + y_i T_i)} (T_0 + \sum_{i=0}^{n} \sum_{\alpha \in \mathfrak{a}} \frac{1}{\alpha!} \langle T_0, \chi^\alpha, \frac{T_n-1}{(q^{-1} - q)} \rangle) \cdot e^{\chi T_i}$$

where $\chi = \sum_{i=0}^{n} y_i T_i$

$$\frac{T_n-1}{q^{-1} - q} = \frac{q T_n-1}{1 - q^4}$$

is expanded in a power series

while $J_p = \sum T_i J_i$

Minor symmetry for $P^n$, then, is:

$$J_i = q_i \quad (i \leq n)$$

Again focus on $dim 2$, though this is not essential.

Fix $M = \mathbb{Z}^2$, $N = M^\vee$. Fix a fan $\Sigma$ in $M_R$, $M_R \rightarrow N_R$. 

Recall that we have
\[ 0 \to K_\Sigma \to T_\Sigma \to N \to 0 \]
true if \( X_\Sigma \) is smooth.

**Def:** A tropical disk is a tropical curve with one univalent vertex without a balancing condition, which is genus 0.

**Ex:** \( V \to \Sigma \to \overline{V} \to \Sigma \to \overline{V} \to \Sigma \to V \to \Sigma \)

These live in \( X_\Sigma \) if all unbounded edges are \( \| \) to \( \Sigma \).

We say a disk \( h: \Gamma \to M_R \) has boundary \( \partial h \) if \( h(V_{out}) = \partial \).

Fix \( Q \supset P_0 \supset P_k \in M_R \) general. Consider a disk which has boundary \( \partial h \) and passes through some subset
\[ \exists P_i \mid i \in I(h) \subseteq \{ P_0, \ldots, P_k \} \]
\[ \Delta(h) = \sum d_p \cdot p, \text{ where } d_p = \# \text{ unbounded edges} \to p \]
\[ r(\Delta(h)) \neq 0 \text{ (b/c of univalent non-balanced)} \]
\[ |\Delta(h)| = \sum d_p \]

The Maslov index (ad hoc def) of \( h \) is
\[ MI(h) = 2(|\Delta(h)| - \# I(h)), \text{ The dimension of the moduli space of such disks is } \frac{MI(h)}{2} - 1 \text{ (this is the case)} \]
Define:
\[ R_k = \frac{\prod\{u_i, \ldots, u_k\}}{\{u_i^2, \ldots, u_k^2\}} \]

In a MLE 2 disk, define
\[ \Delta(h) = M_{\text{null}}(h) \Delta(h) \prod_{i \in I(h)} u_i \]
\[ \Delta(h) \in \mathbb{C}[T_\Sigma]. \]
\[ \Delta(h) \in \mathbb{C}[T_\Sigma]. \]
\[ \text{Define} \quad W_k(Q) = y_0 + \sum_{h, MLE} \text{Mono}(h) \]
\[ \text{e.g. } P^2 \]
\[ W_0(Q) = y_0 + x_0 + x_1 + x_2 \]
\[ W_1(Q) = y_0 + x_0 + x_1 + x_2 + u_1 x_1 x_2 \]
$k=2$:

\[ y_0 + x_0 x_1 + x_2 + u_1 x_0 x_1 + u_2 x_0 x_1 + u_1 u_2 x_0 x_2 \]

0 $\rightarrow$ $k \Sigma$ $\rightarrow$ $T^\Sigma \rightarrow$ $\mu$ $\rightarrow$ $0$

Dualize

0 $\rightarrow$ $N$ $\rightarrow$ $T^\Sigma \rightarrow$ $Pic\ X_\Sigma$ $\rightarrow$ $0$

$\Theta C^*$:

0 $\rightarrow$ $N \otimes C^*$ $\rightarrow$ $T^\Sigma \otimes C^*$ $\rightarrow$ $(Pic\ X_\Sigma) \otimes C^*$ $\rightarrow$ $0$

$\text{Spec } C[[M]] \rightarrow$ $\text{Spec } C[[T_{\Sigma}]] \xrightarrow{\varphi} \text{Spec } C[[K_{\Sigma}]]$

$\varphi^2$: $\varphi = x_0 x_1 x_2$

$W_k$ is a function on $\text{Spec } C[[T_{\Sigma}]] \times \text{Spec } R_k[C[y_0]]$

complex moduli space $\rightarrow$ $\text{Spec } C[[K_{\Sigma}]] \otimes R_k[C[y_0]]$
Let \( \Delta_0 = \frac{1}{(2\pi i)^2} \left[ N \otimes S^1 \right] \)

\[
\sum_{\Delta_0} e \frac{2Wk(Q)}{\Omega} \quad \text{expand}
\]

\[
e^2 y_0 \int_{\Delta_0} e \frac{2(Wk(Q) - y_0)}{\Omega}
\]

only get a contribution to integral if \( Q_{12} \) is constant on a torus.

only contribution come from terms \( z^m \) with \( r(a) = 0 \) (this means constant on \( \text{Spec } \mathbb{C}[z][M] \)).

A term in the expansion looks like

\[
g^0 \prod_{i} \frac{1}{\text{Mon}(h_i)^{v_i}} \text{ with } \sum v_i = v,
\]

\( v_i \neq 0 \) (\( v_i = 1 \) unless \( h_i \) is unmarked).

involves things like: \( 2 \sum v_i \Delta(h_i) \)

contributes if \( r(\sum v_i \Delta(h_i)) = 0 \).

\( \Rightarrow \) the curve obtained by gluing the \( h_i \)'s is balanced at \( Q \). (the power of \( y \) involved at least 1 that disks needed to glue together to make a curve).
So minor symmetry for $P^2$ means.

For $P^2$ with descendants.