Mirror Symmetry

$89 - 90$:

Greene-Grasso

$X = \mathbb{P}^4$ quintic 3-fold

family:

$h^1(X) = 1$, $h^2(X) = 101$

$X_\psi : x_0^5 + \ldots + x_5^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0$

$G = \mathbb{Z}_5^3$ acts on $X_\psi$.

$G = \{ (a_0, \ldots, a_4) \mid a_i \in \mathbb{Z}_5^3, \Sigma a_i = 0 \}$

acts on $\mathbb{P}^4$ by:

$(a_0, \ldots, a_4):$

$(x_0, \ldots, x_4) \mapsto (\frac{1}{5} a_0 x_0, \ldots, \frac{1}{5} a_4 x_4)$

$\frac{1}{5} = e^{2\pi i / 5}$ root of $1$.

$X_\psi \rightarrow X_\psi / G$

crepant resolution,

$h^1(X_\psi) = 101$, $h^2(X_\psi) = 1$ (exchange of $\psi$).

Candelas, de la Ossa, Greene, Parkes (1990):

The number of rational curves on $X$ can be predicted by period integrals on $X_\psi$.

$\int_{X_\psi} \omega_\psi$ for $\alpha \in H^3(X_\psi, \mathbb{C})$
1992: Batyrev — mirror symmetry construction for hypersurfaces in toric varieties.

1994: Kontsevich — Homological Mirror Symmetry Conjecture
(have not been done for CY 3-folds yet, only essentially for K3 varieties)

1996: Givental,Ion, Lim, Yau — Verified the Candelas et al. formulas.

 Strominger-Yau-Zaslow conjecture (SYZ)
First attempt to give geometric description on level of CY's

2001: G-Siebert program
(in some sense, translate hard diff. geo conjecture into alg. geo)

2002: Tropical geometry

Mikhalkin

The tropical semiring is \((\mathbb{R}, \oplus, \odot)\) (max is more standard)

\[ a \oplus b = \min (a, b) \quad \text{in the top.} \]

\[ a \odot b = a + b. \quad \text{in the literature, but} \]

( Semiring b/c no operation \(\ominus\) ).

(more used here)

Tropical polynomial

\[ f(x, y) = \sum_{(i, j) \in S} a_{i,j} x^i y^j = \min \left[ a_{i,j} + ix + jy \right] \]

\(S \subseteq \mathbb{R}^2 \) finite set

\( f: \mathbb{R}^2 \to \mathbb{R} \) piecewise linear.

\(V(f) = \text{critical locus of } f \to \text{set of pts. in } \mathbb{R}^2 \text{ where } f \text{ is not smooth.} \)
\[ f(x) = \min \{ 0, 1+x, 1+y \} = f \]

V(f):

\[ \begin{array}{c}
0 \\
1+x \\
-1, -1 \\
1+y \\
\end{array} \]

\[ V(f) \text{ is a graph with each edge having an integer weight.} \]

How to compute this weight?

\[ \text{Ans:} \]

\[ a(i,j) + x + jy \]

\[ \text{Note that} (i, j) - (i', j') \]

\[ \text{has a constant value} \]

\[ E = \text{edge in } V(f) \text{ on the edge } E. \]

Choose a vector \( \bar{v} \in \mathbb{Z}^2 \)

so that it generates \( \mathbb{Z}^2 / T_E \times \mathbb{Z}^2 \).

Then the weight of the edge \( E \) is \( w(E) = ((i, j) - (i', j')) \cdot \bar{v} \)

(All this does is measure how much \( f \) bends along edge)

\[ \text{Fix } M = \mathbb{Z}^2 \]

\[ N = \text{Hom}(M, \mathbb{Z}) \]

\[ M_R = M \otimes \mathbb{Z} \]

\[ N_R = N \otimes \mathbb{Z} \]

Think of \( f = \sum a_n z^n \), \( f(m) = \min_{n \in S} a_n + \langle m, n \rangle \)

\[ S \subseteq N \text{ is a finite set.} \]

Let \( \Delta S \subseteq N_R \) be the convex hull of \( S \). This is called the Newton polytope of \( f \).
Given $f$, consider

$$\Delta f = \text{lower convex hull of } \{(n, a_n) | n \in S \} \subset \mathbb{R}^* \mathbb{R}.$$ 

**Ex:** $f = 1 \circ \begin{pmatrix} 0 & 0 & x \end{pmatrix} + 1 \circ x^2$

The lower body of $\Delta f$ is the $(2,1)$ graph of a fn $h : \Delta S \to \mathbb{R}$.

In general, for $n \in S$, $h(n) \leq a_n$.

This gives a decomposition of $\Delta S$ into polyhedra which are the images of faces of $\Delta f$ under projection $\mathbb{R}^* \mathbb{R} \to \mathbb{R}$. Call this decomposition $\mathcal{P}$.

We can construct the discrete Legendre transform of $h$, a PL function $h : M_R \to \mathbb{R}$.

First, define a polyhedral decomposition of $M_R$:

for $z \in \mathcal{P}$, $\frac{z}{z}$

$$\frac{z}{z} = 1 - m | m \text{ is the slope of a support hyperplane}$$

for $\frac{z}{z} \in \Delta f$.

$$\{(n, h(n)) | n \in Z\}$$

So above picture goes to:

$$\begin{pmatrix} -1 & 1 \end{pmatrix}$$
$h(m) = \max \left\{ -1 \left< -m, v \right> + v \mid v = h(m) \text{ and } v \in \Delta_5 \right\}$

(As above example, got to cells from considering all possible slopes of support hyperplanes)

Graph of $h(m)$:

$= \min \left\{ h(n) + \left< m, n \right> \mid n \in \Delta_5 \right\} = f$

So the convex locus of $f$ is

$V(f) = \bigcup_{\exists \mathbb{P}} \mathbb{P}$,

$\dim \mathbb{P} = 1$

Ex: $(0, 2)$

$\begin{array}{c}
(0, 1) \\
(0, 0) \end{array}$

$\begin{array}{c}
(-1, 1) \\
(-1, -5) \end{array}$