# 18.996 Special Topic - Spring 2007

## Kontsevich's proof of Witten's conjecture and related topics

## Resume of past lectures

- **Wed, Feb 7th**
- Course overview

- **Fri, Feb 9th**
- $\mathbb{R}$-linear oriented isomorphisms $\mathbb{C} \longrightarrow \mathbb{C}$
- Beltrami differentials $\mu$ and distortion $K$
- Diffeomorphisms between Riemann surfaces: $\mu$ and $K$
- $L^\infty$-norm on $\mu$
- Quasi-conformal maps between Riemann surfaces
- Beltrami equation $(\overline{\partial} - \mu\partial)f = 0$ and deformations of complex structures
- Admissible homeomorphisms (quasi-conformal and differentiable except at isolated points)
- Teichmüller space $\mathcal{T}(S)$ of admissible homeomorphisms $f : S \longrightarrow S'$ as a set
- Teichmüller distance $d_T$ on $\mathcal{T}(S)$
- Holomorphic quadratic differentials $q = q(z)dz^2$
- Beltrami differential $\mu = k|q|/q$ (with $k \in (0, 1)$) associated to a holomorphic quadratic differential $q$
- Standard local coordinate for $q$
- Vertical and horizontal foliations of $q$
- Critical points of $q$

- **Mon, Feb 12th**
- Local structure of trajectories of a quadratic differential $q$
- Flat metric $|q|$ associated to a quadratic differential
- Integrability of $q$ and $L^1$-norm for quadr.differentials
- Coupling between $L^1(S,q)$ and $L^\infty(S,\mu)$
- Degree of quadratic differentials and dimension of $\mathcal{Q}(S)$ (=$\mathbb{C}$-vector space of holomorphic quadratic differentials, for $S$ compact Riemann surface of $g(S) \geq 2$)
- Teichmüller deformations $id : S \longrightarrow S' = (S, q, k)$
- Teichmüller uniqueness theorem (statement only)
- Digression on deformations of complex structures in general: Kodaira-Spencer class and $H^1_{\overline{\partial}}(T^{1,0})$
- Uniformization theorem for Riemann surfaces
- Upper half-plane $\mathbb{H}$, automorphisms of $\mathbb{H}$ and Poincaré metric $\lambda$
- **Wed-Fri, Feb 14-16th**
- The space $\mathrm{Hom}(\pi, \mathrm{SL}_2\mathbb{R})$ of Fuchsian representations, with $\pi = \pi_1(S,p)$, and the Fricke space $\mathfrak{F}_S \subset \mathrm{Hom}(\pi, \mathrm{SL}_2\mathbb{R})$
- The representation space $\mathrm{Rep}(\pi, \mathrm{PSL}_2\mathbb{R})$ and the map $\mathfrak{F}_S \to \mathrm{Rep}(\pi, \mathrm{PSL}_2\mathbb{R})$
- The Teichmüller space $\mathcal{T}(S)$ as a set
- The injective map $\mathcal{T}(S) \hookrightarrow \mathrm{Rep}(\pi, \mathrm{PSL}_2\mathbb{R})$ and the lift $\mathcal{T}(S) \hookrightarrow \mathfrak{F}_S$
- $|\mu(g \circ f)| \leq |\mu(g)|\,|\mu(f)|$ for $S \xrightarrow{f} S' \xrightarrow{g} S''$ and equality holds iff $\arg(\mu(f)) = \arg(\mu(g))$
- $\mathcal{Q}_1(S) = \{q \text{ holomorphic quadratic differential on } S \text{ with } \|q\|_1 < 1\}$
  $\mathcal{B}_1(S) = \{\mu \text{ Beltrami differential on } S \text{ with } \|\mu\|_\infty < 1\}$
- $\Psi_1 : \mathcal{Q}_1(S) \longrightarrow \mathcal{B}_1(S)$ given by $\Psi_1(q) = \dfrac{|q|}{q}\|q\|_1$ is continuous
- Existence, uniqueness and regularity of the solution (Morrey) and regular dependence on the initial data (Ahlfors, Bers) for the Beltrami equation $\overline{\partial} w_\mu = \mu \partial w_\mu$ (for $\mu$ Beltrami differential on $\mathbb{H}$ with $\|\mu\|_\infty < 1$ and $w_\mu : \mathbb{H} \longrightarrow \mathbb{H}$ homeomorphism that fixes $0, 1, \infty$)

- If $S = \mathbb{H}/G$, then $\Psi_2 : \mathcal{B}_1(S) \longrightarrow \mathcal{T}(S) \subset \mathfrak{F}_S$ given by $\Psi_2(\mu) = \mathbb{H}/G_\mu$, where $G_\mu := w_\mu G w_\mu^{-1}$, is continuous
- $\Psi_2$ is surjective so $\mathcal{T}(S)$ is connected
- Teichmüller's uniqueness theorem $\implies$ $\Psi_2 \circ \Psi_1$ is injective
- Invariance of domani $\implies$ $\Psi_2 \circ \Psi_1$ is open
- Surjectivity of $\Psi_2 \circ \Psi_1$
- $(\mathcal{T}(S), d_{Teich})$ is a complete metric space, homeomorphic to $\mathcal{T}(S)$ (with the Fricke topology)
- **Tue, Feb 20th**
- Grötzsch's lemma on $C^1$ maps between rectangles
- Composition of admissible homeomorphisms is admissible
- Homeomorphisms $f : S \to S$ homotopic to the identity: $\ell(f(\gamma)) > \ell(\gamma) - 2M$ (with $M = M(f)$ independent of the path $\gamma$)
- Horizontal stretch $\lambda_{f,q,q'}$ and average horizontal stretch
- Teichmüller's uniqueness theorem
- Universal family of Riemann surfaces over $\mathcal{T}(S)$ and universal property (in the smooth category) of $\mathcal{T}(S)$
- **Wed, Feb 21st**
- Length map $\ell : \pi(S) \times \mathcal{T}(S) \to \mathbb{R}_+$
- Right-angled hexagons and pair of pants: existence and uniqueness
- Cross-ratio in $\overline{\mathbb{R}}$ and distance between geodesics in $\mathbb{H}$
- Klein model and Minkowski model in $\mathbb{R}^{2,1}$: positive $-1$-sphere $S^+_{-1}$
- Geodesic $\gamma \subset \Delta \leftrightarrow$ space-like lines $L = \{\lambda v\} \subset \mathbb{R}^{2,1}$
  Points $p \in \Delta \leftrightarrow$ time-like lines $L' = \{\lambda v'\} \subset \mathbb{R}^{2,1}$
  Horocycles on $\Delta \leftrightarrow$ light-like (positive) vectors in $\mathbb{R}^{2,1}$
  cosh of distance between $\gamma_1$ and $\gamma_2 \leftrightarrow \dfrac{|v_1 \cdot v_2|}{\|v_1\| \, \|v_2\|}$
  cosh of distance between $p$ and $\gamma \leftrightarrow \dfrac{|v \cdot v'|}{\|v\| \, \|v'\|}$
- Maximal systems of curves $\{\alpha_1, \ldots, \alpha_{3g-3}\}$ on $S$ and pair of pants decompositions of $S$

- Fenchel-Nielsen real-analytic global coordinates $\{\ell_1, \tau_1, \ldots, \ell_{3g-3}, \tau_{3g-3}\}$ on $\mathcal{T}(S)$: conventions for the twist parameters
- $\mathrm{FN}_{\{\alpha_i\}} : \mathcal{T}(S) \xrightarrow{\sim} (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$
- **Fri, Feb 23rd**
- picture of the Klein model, Poincaré model and Minkowski model
- Kobayashi pseudo-distance on a complex manifold
- Royden's theorem: the Teichmüller distance is the Kobayashi distance on $\mathcal{T}(S)$
- Mapping class group $\Gamma(S)$ of a compact hyperbolic surface and of a hyperbolic surface of finite volume (=diffeomorphisms classes that preserve each cusp)
- Dehn twists, action on $H_1(S)$, action on $\pi_1(S)$
- representation $\Gamma(S) \longrightarrow \mathrm{Out}(\pi_1(S))$
- the stabilizer of a point $[f : S \longrightarrow S']$ in $\mathcal{T}(S)$ is isomorphism to $\mathrm{Aut}(S')$
- the linear system $L = K_{S'}(p_1 + \cdots + p_n)$ and the embedding $|L^{\otimes 3}| : S' \hookrightarrow \mathbb{P}(H^0(S', L^{\otimes 3})^*)$
- if $S$ is compact hyperbolic or hyperbolic of finite volume, then $\mathrm{Aut}(S)$ is a finite group
- structure of locus of $\mathcal{T}(S)$ of Riemann surfaces with automorphisms (smaller Teichmüller spaces)
- the marked length spectrum and the length spectrum
- Announced for next time: the length spectrum is discrete, the action of $\Gamma(S)$ on $\mathcal{T}(S)$ is discontinuous
- **Mon, Feb 26th**
- if a hyperbolic surface $S$ has a cusp $c$, then the horoball at $c$ of circumference 1 is embedded in $S$
- the length spectrum of a hyperbolic surface is discrete
- the mapping class group $\Gamma(S)$ acts discontinuously on the Teichmüller space $\mathcal{T}(S)$
- construction of the (Dirichlet) fundamental domain in $\mathbb{H}$ under the action of a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$
- a manifold $M$, open subsets $U_i \hookrightarrow M$, atlas $\coprod_i U_i \to M$, glueig data $f_{0,ij} : U_{ij} \hookrightarrow U_i$ and $f_{1,ij} : U_{ij} \hookrightarrow U_j \rightsquigarrow [\coprod_{i,j} U_{ij} \rightrightarrows \coprod_i U_i]$

- manifold $M$, atlas=surjective covering space $M_0 \to M$ ($M_0$ a manifold) and presentation=two maps $M_1 \rightrightarrows M_0$ (with $M_1 = M_0 \times_M M_0$) which are surjective covering spaces (with some extra-property)

- sheaves (and sections of sheaves) on $M$ as sheaves (and sections of sheaves) on $M_0$ that satisfy some glueing conditions (given by $M_1$)

- Example: $M_0$ manifold acted upon by a finite group $G$
  Let $M_1 = M_0 \times G$ and $f_0, f_1 : M_1 \to M_0$ be given by the action of $G$ and by the projection onto $M$
  If $G$ acts freely, then $[\, M_1 \rightrightarrows M_0 \,]$ is the presentation of the manifold $M = M_0/G$
  Otherwise, it is the presentation of an orbifold $[M_0/G]$ (for instance, if $M_0$ is a point, then $[M_0/G]$ represents $BG$ in the category of orbifolds)

- **Wed, Feb 28th**

- small categories, topological categories, morphisms and equivalences

- groupoids and sets; topological groupoids and topological spaces; passage to the coarse space

- orbifold structures on a topological space $M$ and equivalences of orbifold structures

- fiber products of groupoids and of orbifolds

- representability of a (1)-morphism of orbifolds and inertia groups

- Example: if $G$ is a finite group, then $[*] \to [*/G]$ is representable (finite of degree $|G|$) and $[*/G] \to [*]$ is not representable (of degree $1/|G|$)

- smoothness, regularity properties of representable maps of orbifolds

- fundamental group, higher homotopy groups, (co)homology of an orbifold, classifying space of a finite group $[*/G] \simeq BG$

- comparison between (co)homology of an orbifold and of its coarse space

- **Fri, Mar 2nd**

- equivalences of categories and of topological categories, homotopies

- orbisimplicial complexes, topological realization and "essential" topological realization (without degenerate simplices), dimension of an orbisimplicial space

- orbifolds are orbitriangulable

- hyperbolic surfaces (resp. Riemann surfaces) with nodes and markings

- Fenchel-Nielsen coordinates adapted to a nodal hyperbolic surface

- augmented Teichmüller space $\overline{\mathcal{T}}(S)$ of stable curves
- Fenchel-Nielsen charts of $\overline{\mathcal{T}}(S)$
- Example of $\overline{\mathcal{T}}(S)$ for $S$ a punctured torus: $\overline{\mathcal{T}}(S) \cong \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ with the horocyclic topology
- **Mon, Mar 5th**
- complex of curves and strata of the augmented Teichmüller space $\overline{\mathcal{T}}(S)$
- smooth Fenchel-Nielsen structure on $\overline{\mathcal{M}}_{g,n}$
- $\overline{\mathcal{M}}_{g,n}^{\mathbb{C}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{FN}$ is Lipschitz but the inverse is not Hölder (Wolpert)
- canonical sheaf of nodal curves (and on punctured curves), tricanonical embedding
- Hilbert scheme $\mathcal{H}$ of tricanonical curves, dimension count, stable locus, quotient by $\mathrm{PGL}(N)$
- slice of $\mathcal{H}$ transverse to the action of $\mathrm{PGL}(N)$ and complex structure on $\overline{\mathcal{M}}_{g,n}$
- stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type
- boundary maps on $\overline{\mathcal{M}}_{g,n}$
- **Wed, Mar 7th**
- Planned: Kodaira-Spencer map, boundary maps, tautological classes, arc complex

## Calendar

| | | | |
|---|---|---|---|
| 1. | Wed | 2/7 | *First class* |
| 2. | Fri | 2/9 | |
| 3. | Mon | 2/12 | |
| 4. | Wed | 2/14 | |
| 5. | Fri | 2/16 | |
| | Mon | 2/19 | **President's holiday** |
| 6. | Tue | 2/20 | **(Monday schedule)** |
| 7. | Wed | 2/21 | |
| 8. | Fri | 2/23 | |
| 9. | Mon | 2/26 | |
| 10. | Wed | 2/28 | |
| 11. | Fri | 3/2 | |
| 12. | Mon | 3/5 | |
| 13. | Wed | 3/7 | |
| 14. | Fri | 3/9 | |
| 15. | Mon | 3/12 | |
| 16. | Wed | 3/14 | |
| 17. | Fri | 3/16 | |
| 18. | Mon | 3/19 | |
| 19. | Wed | 3/21 | |
| 20. | Fri | 3/23 | |
| | MWF | 3/26-30 | **Spring vacation** |
| 21. | Mon | 4/2 | |
| 22. | Wed | 4/4 | |
| 23. | Fri | 4/6 | |
| 24. | Mon | 4/9 | |
| 25. | Wed | 4/11 | |
| 26. | Fri | 4/13 | |
| | Mon | 4/16 | **Patriots Day** |
| 27. | Wed | 4/18 | |
| 28. | Fri | 4/20 | |
| 29. | Mon | 4/23 | |
| 30. | Wed | 4/25 | |
| 31. | Fri | 4/27 | |
| 32. | Mon | 4/30 | |
| 33. | Wed | 5/2 | |
| 34. | Fri | 5/4 | |
| 35. | Mon | 5/7 | |
| 36. | Wed | 5/9 | |
| 37. | Fri | 5/11 | |
| 38. | Mon | 5/14 | |
| 39. | Wed | 5/16 | *Last class* |