18.700 - Fall 2006 - Problem Set 8 (65 points)

(Due on **Tuesday, Dec 12th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (24 points: 2+4+5+5+5+3)
Let $V = \mathbb{R}[t]_{\leq 3}$ be the vector space of real polynomials of degree at most 3 and let $b, \tilde{b} : V \times V \longrightarrow \mathbb{R}$ be two maps defined as $b(p(t), q(t)) = \int_0^1 p(t)q(t)dt$ and $\tilde{b}(p(t), q(t)) = \int_0^1 p(t)q'(t)dt$.

(a) Show that b and $\tilde{b}$ are bilinear forms.

(b) Determine whether b is symmetric; if so, whether it's positive-definite. The same for $\tilde{b}$.

(c) Let $\mathcal{C} = \{v_0, v_1, v_2, v_3\}$ with $v_i = t^i$. Determine the matrices that represent b and $\tilde{b}$ with respect to the basis $\mathcal{C}$.

(d) Is b nondegenerate? If not, determine the kernel of the induced map $L_b : V \longrightarrow V^*$, defined as $L_b(v) = b(v, \cdot)$. The same for $\tilde{b}$.

(e) If b is symmetric and nondegenerate, determine an orthonormal basis for b. The same for $\tilde{b}$.

(f) For those b and/or $\tilde{b}$ which are symmetric: determine the subspace of V which is orthogonal to the subspace $W := \text{span}\{t\} \subset V$.

Problem 2. (14 points: 2+6+2+4)

Let $B = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -1 & 2 & -2 & 3 & 0 \\ 0 & 1 & -2 & 1 & 2 \end{pmatrix}$ represent a homomorphism $\mathbb{R}^5 \longrightarrow \mathbb{R}^5$.

(a) Compute the characteristic polynomial $p_B(t)$ and say wheter B is triangularizable.

(b) Determine $\dim \ker(B - eI)^k$ for every $k > 0$ and every root e of $p_B(t)$ and determine the minimal polynomial of B.

(c) If B is triangularizable, determine its Jordan form. Otherwise, just find the primary decomposition.

(d) If B is triangularizable, determine a Jordan basis for B. Otherwise, determine a basis adapted to the primary decomposition.

Problem 3. (14 points: 7+7)
Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a symmetric matrix, which is positive-definite (i.e. the associated inner product on $\mathbb{R}^n$ is positive-definite).

(a) Show that there exists a positive-definite symmetric matrix $R \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $R^2 = A$. (Hint: first solve the case of A diagonal.) **CANCELLED! You can solve part (b) even without using part (a).**

(b) Let $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a symmetric matrix, which is negative-definite (i.e. $-B$ is positive-definite). Show that $\mathrm{tr}(AB) < 0$.

Problem 4. (20 points: 5+5+6+4)
Let $b, \tilde{b} : V \times V \longrightarrow \mathbb{R}$ symmetric bilinear forms defined on the $\mathbb{R}$-vector space $V = \mathcal{M}_{n\times n}(\mathbb{R})$ as $b(X,Y) = \mathrm{tr}(XY)$ and $\tilde{b}(X,Y) = \mathrm{tr}(X^tY)$.

(a) Determine whether b is degenerate: if so, compute its radical $\mathrm{Rad}(b) = \ker(L_b)$, where $L_b : V \longrightarrow V^*$. The same for $\tilde{b}$.

(b) Determine whether b (resp. $\tilde{b}$) is positive-definite.

(c) Let $\mathcal{S} \subset V$ be the subspace of symmetric matrices, $\mathcal{A} \subset V$ the subspace of skew-symmetric matrices and $\mathcal{U} \subset V$ be the subspace of strictly upper triangular matrices. Show that the restriction of b to $\mathcal{S}$ is positive-definite, the restriction of b to $\mathcal{A}$ is negative-definite and the restriction of b to $\mathcal{U}$ is zero (i.e. $\mathcal{U}$ is an *isotropic subspace*).

(d) Prove that $\mathcal{U}$ is a maximal isotropic subspace for b (i.e. that there is no other isotropic subspace of dimension strictly greater than that of $\mathcal{U}$).