18.700 - Fall 2006 - Solutions to Problem Set 8

Problem 1.

(a) For every $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $p, q, r, s \in V$, we have $b(\alpha p + \beta q, \gamma r + \delta s) = \int_0^1 (\alpha p + \beta q)(\gamma r + \delta s)dt = \alpha\gamma \int_0^1 pr\,dt + \alpha\delta \int_0^1 ps\,dt + \beta\gamma \int_0^1 qr\,dt + \beta\delta \int_0^1 qs\,dt = \alpha\gamma b(p, r) + \alpha\delta b(p, s) + \beta\gamma b(q, r) + \beta\delta b(q, s)$, which shows that b is bilinear.
Moreover, $\tilde{b}(\alpha p + \beta q, \gamma r + \delta s) = \int_0^1 (\alpha p + \beta q)(\gamma r' + \delta s')dt = \alpha\gamma \int_0^1 pr'\,dt + \alpha\delta \int_0^1 ps'\,dt + \beta\gamma \int_0^1 qr'\,dt + \beta\delta \int_0^1 qs'\,dt = \alpha\gamma\tilde{b}(p, r) + \alpha\delta\tilde{b}(p, s) + \beta\gamma\tilde{b}(q, r) + \beta\delta\tilde{b}(q, s)$, so that $\tilde{b}$ is bilinear too.

(b) b is symmetric. In fact, $b(p, q) = \int_0^1 pq\,dt = \int_0^1 qp\,dt = b(q, p)$ for every $p, q \in V$.

$\tilde{b}$ is not symmetric. In fact $\tilde{b}(1, t) = \int_0^1 1(t)'dt = \int_0^1 dt = 1$ but $\tilde{b}(t, 1) = \int_0^1 t(1)'dt = 0$.

b is positive-definite. In fact, if $p \in V$, then $b(p, p) = \int_0^1 p^2 dt \geq 0$ and $b(p, p) = 0$ if and only if $p = 0$.

(c) Observe that $\int_0^1 t^k dt = \frac{1}{k+1}$, hence $b(v_i, v_j) = \int_0^1 t^i t^j dt = \frac{1}{i+j+1}$, so that

$$M_{\mathcal{C}}(b) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$$

Instead, $\tilde{b}(v_i, v_j) = j \int_0^1 t^i t^{j-1} = \frac{j}{i+j}$, so that

$$M_{\mathcal{C}}(\tilde{b}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1/2 & 2/3 & 3/4 \\ 0 & 1/3 & 1/2 & 3/5 \\ 0 & 1/4 & 2/5 & 1/2 \end{pmatrix}$$

(d) Yes, b is nondegenerate, because it is positive-definite, hence $\ker(L_b) = \{0\}$.
Instead, $\tilde{b}$ is degenerate (even though not symmetric), because the first column of $M_{\mathcal{C}}(\tilde{b})$ is zero. Exploiting the fact that ${}^t M_{\mathcal{C}}(\tilde{b}) = M_{\mathcal{C}^*}^{\mathcal{C}}(L_{\tilde{b}})$, one can check that $\ker(L_{\tilde{b}})$ is spanned by the vector $20t^3 - 30t^2 + 12t - 1$.

(e) Applying Gram-Schmidt to the basis $\mathcal{C}$ we find the orthonormal basis: $\{w_0, w_1, w_2, w_3\}$, where $w_0 = v_0 = 1$, $w_1 = \sqrt{3}(2t - 1)$, $w_2 = \sqrt{5}(6t^2 - 6t + 1)$ and $w_3 = \sqrt{7}(20t^3 - 30t^2 + 12t - 1)$.
The explicit computation is the following. Define $u_0 := v_0$ and $w_0 := \frac{u_0}{\sqrt{b(u_0, u_0)}} = v_0 = 1$, because $b(1, 1) = 1$.

Then define $u_1 := v_1 - b(v_1, w_0)w_0 = t - b(1,t)1 = t - 1/2$ and then set $w_1 := \dfrac{u_1}{\sqrt{b(u_1,u_1)}} = \sqrt{3}(2t-1)$, because $b(u_1,u_1) = 1/12$.
Define $u_2 := v_2 - b(v_2,w_0)w_0 - b(v_2,w_1)w_1 = t^2 - b(t^2,1)1 - b(t^2,\sqrt{3}(2t-1))\sqrt{3}(2t-1) = t^2 - 1/3 - 3(2t-1)/6 = t^2 - t + 1/6$. Then $w_2 := \dfrac{u_2}{\sqrt{b(u_2,u_2)}} = \sqrt{5}(6t^2-6t+1)$, because $b(u_2,u_2) = 1/180$.
Finally, define $u_3 := v_3 - b(v_3,w_0)w_0 - b(v_3,w_1)w_1 - b(v_3,w_2)w_2 = t^3 - b(t^3,1)1 - b(t^3,\sqrt{3}(2t-1))\sqrt{3}(2t-1) - b(t^3,\sqrt{5}(6t^2-6t+1))\sqrt{5}(6t^2-6t+1) = t^3 - 3t^2/2 + 3t/5 - 1/20$ and $w_3 := \dfrac{u_3}{\sqrt{b(u_3,u_3)}} = \sqrt{7}(20t^3 - 30t^2 + 12t - 1)$, because $b(u_3,u_3) = 1/2800$.

(f) $W^\perp = \{p \in V \mid b(p,t) = \int_0^1 pt\,dt = 0\}$.
If $p = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, then $\int_0^1 pt = a_0/2 + a_1/3 + a_2/4 + a_3/5$, so that $W^\perp = \{p = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in V \mid a_0/2 + a_1/3 + a_2/4 + a_3/5 = 0\}$.
A basis for $W^\perp$ is $\{3-2t,\ 4-2t^2,\ 5-2t^3\}$.

Problem 2.

Let $B = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -1 & 2 & -2 & 3 & 0 \\ 0 & 1 & -2 & 1 & 2 \end{pmatrix}$ represent a homomorphism $\mathbb{R}^5 \longrightarrow \mathbb{R}^5$.

(a) The characteristic polynomial is $p_B(t) = (3-t)^3(2-t)^2$, so that B is triangularizable.

(b) $\dim\ker(B-2I) = 1$ (and it is spanned by e_5). Necessarily $\dim\ker(B-2I)^{=}2$ (and it is spanned by e_5 and $e_1 + e_4$).
$\dim\ker(B-3I) = 2$ (and it is spanned by $e_4 + e_5$ and $2e_2 + 2e_3 + e_4 - e_5$). Necessarily $\dim\ker(B-3I)^2 = 3$ (and it is spanned by e_4+e_5, $2e_2+2e_3+e_4-e_5$ and e_1+e_2).
So the minimal polynomial is $p_{B,min}(t) = (t-2)^2(t-3)^2$.

(c) Corresponding to the eigenvalue 2 there is one block of size 2, whereas corresponding to the eigenvalue 2 there are two blocks, one of size 2 and one of size 1. Hence, the Jordan form of B is

$$J := \left(\begin{array}{cc|cc|c} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

(d) Call $x_2 = e_1 + e_4$ and $x_1 = (B-2I)x_2 = e_5$, so that $(B-2I)x_1 = 0$.
Call $y_2 = e_1 + e_2$ and $y_1 = (B-3I)y_2 = e_4 + e_5$, so that $(B-3I)y_2 = 0$.
Finally, call $z_1 = 2e_2 + 2e_3 + e_4 - e_5$ and let $\mathcal{B} = \{x_2, x_1, y_2, y_1, z_1\}$.
Then $M_{\mathcal{B}}^{\mathcal{B}}(B) = J$ and so $\mathcal{B}$ is a Jordan basis for B.

Problem 3.
Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a symmetric matrix, which is positive-definite (i.e. the associated inner product on $\mathbb{R}^n$ is positive-definite).

(a) **This part was cancelled!**
However, the solution is the following. A is symmetric and positive-definite. By the spectral theorem, there exists $N \in O(n, \mathbb{R})$ such that ${}^tNAN = N^{-1}AN = D$, with D diagonal with $D_{ii} > 0$.
Let E be a diagonal matrix with $E_{ii} = \sqrt{D_{ii}} > 0$.
Then we can define $R = NE\,{}^tN = NEN^{-1}$, which is clearly symmetric and positive-definite.
Moreover, $R^2 = NEN^{-1}NEN^{-1} = NE^2N^{-1} = NDN^{-1} = A$.

(b) **In this solution, we do not use part (a)!**
First remark that: if P is a symmetric negative-definite matrix, then $\mathrm{tr}(P) = \sum_{i=1}^{n} P_{ii} < 0$, because $P_{ii} < 0$.
As A is symmetric, we can find an $N \in \mathrm{GL}(n, \mathbb{R})$ such that $NA\,{}^tN = D$ is diagonal (we do not assume that N is an orthonal matrix!). Clearly, $D_{ii} > 0$, because A is positive-definite, and we can still define a diagonal E such that $E_{ii} = \sqrt{D_{ii}} > 0$. So we can write $A = ME^2\,{}^tM$, with $M = N^{-1}$.
Now, $\mathrm{tr}(AB) = \mathrm{tr}(ME^2\,{}^tMB) = \mathrm{tr}(E\,{}^tMBME) = \mathrm{tr}({}^t(ME)B(ME)) < 0$, because $P = {}^t(ME)B(ME)$ is symmetric and congruent to B, so P represents a negative-definite scalar product

Problem 4.
Let $b, \tilde{b} : V \times V \longrightarrow \mathbb{R}$ symmetric bilinear forms defined on the $\mathbb{R}$-vector space $V = \mathcal{M}_{n\times n}(\mathbb{R})$ as $b(X, Y) = \mathrm{tr}(XY)$ and $\tilde{b}(X, Y) = \mathrm{tr}(X^tY)$.

(a) From part (b), the bilinear form $\tilde{b}$ is positive-definite and so nondegenerate.
From part (c), the restriction of b to $\mathcal{A}_n$ is negative-definite and the restriction of b to $\mathcal{S}_n$ is positive-definite.
We have the decomposition $\mathcal{M}_{n\times n} = \mathcal{A}_n \oplus \mathcal{S}_n$, because $\dim \mathcal{A}_n + \dim \mathcal{S}_n = n(n-1)/2 + n(n+1)/2 = n^2 = \dim \mathcal{M}_{n\times n}$ and $\mathcal{A}_n \cap \mathcal{S}_n = \emptyset$ (because if $M \in \mathcal{S}_n \cap \mathcal{A}_n$, then ${}^tM = M$ and ${}^tM = -M$, so that $M = -M \implies M = 0$). Another argument to prove this decomposition is that every matrix M can be written as $\dfrac{M + {}^tM}{2} + \dfrac{M - {}^tM}{2}$, with $\dfrac{M + {}^tM}{2}$ symmetric and $\dfrac{M - {}^tM}{2}$ skew-symmetric.
Anyway, part (c) shows that the positivity of b (the maximal dimension of a subspace where the restriction of b is positive-definite; equivalently, the number of $+1$'s appearing in Sylvester's form of b) is at least $n(n+1)/2$ and the negativity is at least $n(n-1)/2$. Moreover $\mathcal{A}_n$ and $\mathcal{S}_n$ are orthogonal with respect to b (and also $\tilde{b}$), because for $A \in \mathcal{A}_n$ and $S \in \mathcal{S}_n$ we have $b(A, S) = \mathrm{tr}(AS) = \mathrm{tr}({}^t(AS)) = \mathrm{tr}({}^tS\,{}^tA) = -\mathrm{tr}(SA) = -\mathrm{tr}(AS)$ and so $b(A, S) = 0$. Hence, if $M \in \mathrm{Rad}(b)$, write $M = A + S$, with $A \in \mathcal{A}_n$ and $S \in \mathcal{S}_n$. Then for every $A' \in \mathcal{A}_n$, $0 = b(M, A') = b(A, A') + b(S, A') = b(A, A')$, which implies that $A = 0$.
Similarly, for every $S' \in \mathcal{S}$ we have $0 = b(M, S') = b(A, S') + b(S, S') = b(S, S')$, which implies that $S = 0$. Hence, $M = 0$ and $\mathrm{Rad} b = \{0\}$.

(b) $\tilde{b}$ is positive-definite. In fact, $\tilde{b}(M,M) = \mathrm{tr}({}^tMM) = \sum_{i,j=1}^{n} (M_{ij})^2 \geq 0$, and its equal to zero if and only if $M = 0$.
Instead, b is not positive-definite (if $n > 1$!), because the matrix M made all of zeroes except $M_{2,1} = 1$ has the property that $M^2 = 0$ and so $b(M,M) = \mathrm{tr}(M^2) = 0$.

(c) If $0 \neq S \in \mathcal{S}_n$ is symmetric, then $b(S,S) = \mathrm{tr}(SS) = \mathrm{tr}({}^tSS) = \tilde{b}(S,S) > 0$. So the restriction of b to $\mathcal{S}_n$ is positive-definite.
If $0 \neq A \in \mathcal{A}_n$ is skew-symmetric, then $b(A,A) = \mathrm{tr}(AA) = \mathrm{tr}(-{}^tAA) = -\tilde{b}(A,A) < 0$. So the restriction of b to $\mathcal{A}_n$ is negative-definite.
If $U, V \in \mathcal{U}_n$ are strictly upper-triangular, then UV is also strictly upper-triangular. Hence, $b(U,V) = \mathrm{tr}(UV) = 0$ and so the restriction of b to $\mathcal{U}_n$ is zero, i.e. $\mathcal{U}_n$ is isotropic for b.

(d) By (c), the signature of b is $(n(n+1)/2, n(n-1)/2)$, so that a maximal isotropic subspace has dimension $= \min\{n(n+1)/2, n(n-1)/2\} = n(n-1)/2$.
In fact, $\mathcal{U}_n$ is isotropic and $\dim \mathcal{U}_n = n(n-1)/2$, so that it is maximal isotropic.