18.700 - Fall 2006 - Problem Set 7 (60 points)

(Due on **Tuesday, Nov 28th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (16 points: 6+10)
Let $f : V \longrightarrow V$ be an endomorphism of a vector space V of dimension n over $\mathbb{F}$. Let $p_f(t) = r_1(t) \cdots r_k(t)$ be its characteristic polynomial, with r_i irreducible over $\mathbb{F}$ and $(r_i, r_j) = 1$ for $i \neq j$ (i.e. the r_i's are pairwise coprime).

(a) Suppose $k = 1$ and let $0 \neq v \in \ker(r_1(f)) = V$. Prove that $\mathcal{B} = \{v, f(v), f^2(v), \ldots, f^{n-1}(v)\}$ is a set of linearly independent vectors of V and so a basis.

(b) Suppose $k \geq 1$, so that $V = \ker(r_1(f)) \oplus \ker(r_2(f)) \oplus \cdots \oplus \ker(r_k(f))$ is the primary decomposition, and let $0 \neq v_i \in \ker(r_i(f))$ for every $i = 1, \ldots, k$. Define $v := v_1 + v_2 + \cdots + v_k$. Prove that $\mathcal{B} = \{v, f(v), f^2(v), \ldots, f^{n-1}(v)\}$ is a set of linearly independent vectors of V and so a basis.

Problem 2. (20 points: 4+8+2+4+2)

Let $A_c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & c & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & c & -1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}$ be a matrix in $\mathcal{M}_{5\times 5}(\mathbb{R})$.

(a) Compute the characteristic polynomial $p_{A_c}(t)$ of A_c for every value of $c \in \mathbb{R}$.

(b) For every irreducible polynomial $q(t)$ that divides $p_{A_c}(t)$ and every $k \geq 1$, compute $\dim \ker(q(f)^k)$.

(c) Compute the minimal polynomial $p_{A_c,min}(t)$ of A_c.

(d) For which values $c \in \mathbb{R}$ is A_c triangularizable or diagonalizable over $\mathbb{R}$?

(e) For which values $c \in \mathbb{C}$ is A_c diagonalizable?

Problem 3. (10 points)
Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a diagonalizable matrix. Prove that: $\exists k \in \mathbb{N}$ such that $A^k = I \iff A^2 = I$.

Problem 4. (14 points)
Classify all the 2×2 matrices A with rational coefficients such that $A^6 = I$ up to similitude.
(The text means: "Let $\mathcal{X} \subset \mathcal{M}_{2\times 2}(\mathbb{Q})$ be the subset of matrices A such that $A^6 = I$. Say that $A, B \in \mathcal{X}$ are equivalent if A is similar to B. How many equivalence classes are there in $\mathcal{X}$?")