18.700 - Fall 2006 - Solutions to Problem Set 7

(Due on **Tuesday, Nov 28th**)

Problem 1.

(a) Let $\sum_{i=0}^{n-1} a_i f^i(v) = 0$ for $a_i \in \mathbb{F}$. Then $q(f)(v) = 0$ where $q(t) = \sum_{i=0}^{n-1} a_i t^i \in \mathbb{F}[t]$. We want to show that $q(t) = 0$. By contradiction, assume $q(t) \neq 0$ and so $\deg(q(t)) \leq n-1$. As $r_1(t)$ has degree n and is irreducible, then $q(t)$ and $r_1(t)$ are coprime. Hence, there exist polynomials $\alpha(t), \beta(t) \in \mathbb{F}[t]$ such that $\alpha(t)q(t) + \beta(t)r_1(t) = 1$. Evaluating at f, we get $\alpha(f)q(f) + \beta(f)r_1(f) = I$. Evaluating at v, we get $\alpha(f)q(f)(v) + \beta(f)r_1(f)v = v$. By Cayley-Hamilton, $r_1(f) = 0$, so $\alpha(f)[q(f)(v)] = v \neq 0$, so that $q(f)(v) \neq 0$. Contradiction!

(b) Notice that $\ker(r_i(f))$ is f-invariant and so $q(f)(v_i) \in \ker(r_i(f))$ for every $q(t) \in \mathbb{F}[t]$. Hence, because the $\ker(r_i(f))$'s are in direct sum, $q(f)(v) = 0$ if and only if $q(f)(v_i) = 0$ for every $i = 1, \ldots, k$. Call f_i the restriction of f to $\ker(r_i(f))$. Notice that $r_i(f_i) = 0$ and $\ker(r_i(f)) \neq \{0\}$, so that the minimal polynomial of f_i is exactly $r_i(t)$ (because it must have degree at least 1 and it must divide $r_i(t)$, but $r_i(t)$ is irreducible).

Now we mimick the argument of (a) to prove (b), namely let $\sum_{i=0}^{n-1} a_i f^i(v) = 0$ for $a_i \in \mathbb{F}$. Then $q(f)(v) = 0$ and so $q(f)(v_j) = 0$ for every $j = 1, \ldots, k$, where $q(t) = \sum_{i=0}^{n-1} a_i t^i \in \mathbb{F}[t]$. We want to show that $q(t) = 0$. By contradiction, assume $q(t) \neq 0$ and so $\deg(q(t)) \leq n-1$. For every $j = 1, \ldots, k$ we have that $q(f)(v_j) = 0$ and so $q(t)$ is divisible by $r_j(t)$ (because $r_j(t)$ is irreducible). [Otherwise $q(t)$ and $r_i(t)$ would be coprime and so there would exist polynomials $\alpha(t), \beta(t) \in \mathbb{F}[t]$ such that $\alpha(t)q(t) + \beta(t)r_i(t) = 1$. Evaluating at f_i, we would get $\alpha(f_i)q(f_i) + \beta(f_i)r_i(f_i) = I$. Evaluating at v_i, we get $\alpha(f_i)q(f_i)(v_i) + \beta(f_i)r_i(f_i)v_i = v_i$. As $r_i(f_i) = 0$, we obtain $\alpha(f_i)[q(f_i)(v_i)] = v_i \neq 0$, so that $q(f_i)(v_i) \neq 0$. Contradiction!] Hence, $q(t)$ must be divisible by $r_1(t), r_2(t), \ldots, r_k(t)$. As the $r_i(t)$'s are pairwise coprime, then $q(t)$ must be divisible by $r_1(t)r_2(t)\cdots r_k(t) = p_f(t)$. This is a contradiction, because $p_f(t)$ has degree n whereas $q(t)$ has degree at most $n-1$.

Problem 2.

Let $A_c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & c & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & c & -1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}$ be a matrix in $\mathcal{M}_{5\times 5}(\mathbb{R})$.

(a) $p_{A_c}(t) = (2-t)(c-t)(1-t)(t^2 - ct + 1)$.

(b) If $\boxed{c \in (-2,2) \text{ but } c \neq 1}$, then $(c-t) \neq (1-t)$ and $(t^2 - ct + 1)$ is irreducible (over $\mathbb{R}$). Hence, $\dim\ker(2I - A_c) = \dim\ker(I - A_c) = \dim\ker(cI - A_c) = 1$ and $\dim\ker(A_c^2 - cA_c + I) = 2$.

If $\boxed{c=1}$, then (t^2-ct+1) is irreducible. Direct computation (it's sufficient to analyze the restriction $A_{1,W}$ of A_1 to the invariant subspace $W=\operatorname{span}\{e_2,e_3\}$, because $p_{A_{1,W}}(t)=(1-t)^2$) gives $\mu_{geom,A_1}(1)=\dim\ker(I-A_1)=1<2=\dim\ker(I-A_1)^2=\mu_{alg,A_1}(1)$. Moreover, $\dim\ker(2I-A_1)=1$ and $\dim\ker(A_1^2-A_1+I)=2$.

If $\boxed{c=-2}$, then $(t^2-ct+1)=(t+1)^2$. Analyzing the restriction $A_{-2,Z}$ of A to the invariant subspace $Z=\operatorname{span}\{e_4,e_5\}$ (because $p_{A_{-2,Z}}(t)=(t+1)^2$), we immediately see that $\dim\ker(I+A_{-2})=1<2=\dim\ker(I+A_{-2})^2$. Moreover, $\dim\ker(2I+A_{-2})=\dim\ker(-2I+A_{-2})=\dim\ker(I-A_{-2})=1$.

If $\boxed{c=2}$, then $(t^2-ct+1)=(t-1)^2$. Analyzing the restriction $A_{2,V}$ of A to the invariant subspace $V=\operatorname{span}\{e_3,e_4,e_5\}$ (because $p_{A_{2,V}}(t)=(t+1)^3$), we immediately see that $\dim\ker(I-A_2)=2<3=\dim\ker(I-A_2)^2$. Moreover, by direct computation we find $\dim\ker(2I-A_2)=2$.

For $\boxed{c<-2 \text{ or } c>2 \text{ but } c\neq 5/2}$, we have $\dim\ker(2I-A_c)=\dim\ker(I-A_c)=\dim\ker(cI-A_c)=1$ and $\dim\ker(A_c^2-cA_c+I)=2$.

If $\boxed{c=5/2}$, then $p_{A_{5/2}}(t)=-(t-2)^2(t-1)(t-1/2)(t-5/2)$. Direct computation gives $\dim\ker(2I-A_{5/2})=1$, whereas $\dim\ker(I-A_{5/2})=\dim\ker(\frac{1}{2}I-A_{5/2})=\dim\ker(\frac{5}{2}I-A_{5/2})=1$.

(c) For $\boxed{c\neq 2}$, we have $p_{min,A_c}(t)=-p_{A_c}(t)$.
Instead, for $\boxed{c=2}$, we have $p_{min,A_2}(t)=(t-2)(t-1)^2$ whereas $p_{A_2}(t)=-(t-2)^2(t-1)^3$.

(d) A_c is $\boxed{\text{triangularizable over } \mathbb{R}}$ for $c\in\mathbb{R}$ whenever $p_{A_c}(t)$ is completely factorizable over $\mathbb{R}$, i.e. if $\boxed{c\leq -2 \text{ or } c\geq 2}$.
A_c is $\boxed{\text{diagonalizable over } \mathbb{R}}$ for $c\in\mathbb{R}$ whenever $p_{min,A_c}(t)$ is completely factorizable over $\mathbb{R}$ and has distinct roots. This happens when $\boxed{c<-2 \text{ or } c>2 \text{ but } c\neq 5/2}$.

(e) A_c is $\boxed{\text{diagonalizable over } \mathbb{C}}$ for $c\in\mathbb{C}$ whenever $p_{min,A_c}(t)$ has distinct roots. This happens when $\boxed{c\in\mathbb{C}\setminus\{1,2,-2,5/2\}}$.

Problem 3.

As A is diagonalizable, then $p_{A,min}(t)=(t-e_1)\cdots(t-e_m)$ with $e_i\neq e_j$ for $i\neq j$, for some $1\leq m\leq n$ and $e_i\in\mathbb{R}$.
As $A^k=I$, then $p_{A,min}(t)$ divides (t^k-1), which means that $e_i^k=1$ for every i. This implies that $e_i=\pm1$ (because $e_i\in\mathbb{R}$). Hence, $p_{A,min}(t)$ can be $(t-1)$ or $(t+1)$ or $(t-1)(t+1)$. In any case, $p_{A,min}(t)$ divides (t^2-1), that is $A^2=I$.

Problem 4.

As $A^6=I$, then the minimal polynomial $p_{A,min}(t)$ divides $(t^6-1)=(t-1)(t+1)(t^2+t+1)(t^2-t-1)$. Moreover, $p_{A,min}(t)$ has degree 1 or 2. There are five cases.
If $\boxed{p_{A,min}(t)=t-1}$, then $A=I_2$.
If $\boxed{p_{A,min}(t)=t+1}$, then $A=-I_2$.

If $\boxed{p_{A,min}(t) = (t-1)(t+1)}$, then A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If $\boxed{p_{A,min}(t) = t^2 + t + 1}$, take a nonzero vector $0 \neq v_1 \in \mathbb{Q}^2$ and call $v_2 := A(v_1)$. Because A has no eigenvectors, then $\mathcal{B} = \{v_1, v_2\}$ is a basis of $\mathbb{Q}^2$ and we have that A is similar to $M_{\mathcal{B}}^{\mathcal{B}}(A) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ because $A(v_1) = v_2$ and $A(v_2) = A^2(v_1) = (-A - I)(v_1) = -v_1 - v_2$.

If $\boxed{p_{A,min}(t) = t^2 - t + 1}$, take a nonzero vector $0 \neq w_1 \in \mathbb{Q}^2$ and call $w_2 := A(w_1)$. Because A has no eigenvectors, then $\mathcal{C} = \{w_1, w_2\}$ is a basis of $\mathbb{Q}^2$ and we have that A is similar to $M_{\mathcal{C}}^{\mathcal{C}}(A) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ because $A(w_1) = w_2$ and $A(w_2) = A^2(w_1) = (A - I)(w_1) = -w_1 + w_2$.

Hence, we only have the five classes above.