18.700 - Fall 2006 - Problem Set 6 (60 points)

(Due on **Tuesday, Nov 7th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (14 points)

Let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ be a matrix of rank 1.
Prove that A is diagonalizable if and only if $\mathrm{tr}(A) \neq 0$.

Problem 2. (20 points: 2+5+4+5+4)

Let $A_c = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & c & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ be a matrix in $\mathcal{M}_{4\times 4}(\mathbb{Q})$.

(a) Compute the characteristic polynomial of A_c for every value of $c \in \mathbb{Q}$.

(b) For every value of $c \in \mathbb{Q}$, determine the eigenvalues and the eigenspaces of A_c.

(c) For which values of c is A_c diagonalizable?

(d) Is there a value of $c \in \mathbb{Q}$ such that A_c is similar to $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$?

(e) Let B^t be the transpose of B. Is $B - B^t$ similar to an upper triangular matrix?

Problem 3. (10 points: 5+5)

Let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ be an $n \times n$ matrix.
A *minor* B of A *of order* k is a $k \times k$ matrix obtained from A by choosing k rows and k columns (and discarding all the other rows and columns).

(a) Show that, if there is a minor of A of order k which is invertible, then the rank of A is at least k.

(b) Show that, if the rank of A is k and B is the minor of order k obtained from A by selecting k linearly independent rows and k linearly independent columns, then B is inverible.

Problem 4. (16 points: 4+3+6+3)

(a) Let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ and $B \in \mathcal{M}_{m\times m}(\mathbb{F})$.
Express the characteristic polynomial of $M = \left(\begin{array}{c|c} A_{n\times n} & 0_{n\times m} \\ \hline 0_{m\times n} & B_{m\times m} \end{array}\right) \in \mathcal{M}_{(m+n)\times(m+n)}(\mathbb{F})$ as a function of the characteristic polynomials of A and B.

(b) Prove that M is diagonalizable if and only if A and B are both diagonalizable.

(c) Let C be an $n\times m$ matrix. Express the characteristic polynomial of $N = \left(\begin{array}{c|c} A_{n\times n} & C_{n\times m} \\ \hline 0_{m\times n} & B_{m\times m} \end{array}\right) \in \mathcal{M}_{(m+n)\times(m+n)}(\mathbb{F})$ as a function of the characteristic polynomials of A, B and C.

(d) Assume A and B diagonalizable.
Prove or disprove: N is diagonalizable if and only if $C = 0$.